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Nonhyperbolic reflection moveout of $P$-waves:
An overview and comparison of reasons

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ABSTRACT

The familiar hyperbolic approximation of $P$-wave reflection moveout is exact for homogeneous isotropic or elliptically anisotropic media above a planar reflector. Any realistic combination of heterogeneity, reflector curvature, and nonelliptic anisotropy will cause departures from hyperbolic moveout at large offsets. Here, we analyze the similarities and differences in the influence of those three factors on $P$-wave reflection traveltimes. Using the weak-anisotropy approximation for velocities in transversely isotropic media with a vertical symmetry axis (VTI model), we show that although the nonhyperbolic moveout due to both vertical heterogeneity and reflector curvature can be interpreted in terms of effective anisotropy, such anisotropy is inherently different from that of a generic homogeneous VTI model.

INTRODUCTION

The hyperbolic approximation of $P$-wave reflection traveltimes in common-midpoint gathers plays an important role in conventional seismic data processing and interpretation. It is well known that hyperbolic moveout gives exact traveltimes for homogeneous isotropic or elliptically anisotropic media overlaying a plane dipping reflector. Deviations from this simple model generally cause departure from hyperbolic moveout. If the nonhyperbolicity is measurable, we can take it into account to correct errors in conventional processing or to obtain additional information about the medium. To achieve this, however, it is important to know what causes the $P$-wave moveouts to be nonhyperbolic. Although seismic anisotropy is one possible reason, it is not always the dominant one; others include the vertical or lateral heterogeneity and reflector curvature. In this paper, we give a theoretical description of $P$-wave reflection traveltimes in different models and compare the behavior and degree of nonhyperbolic moveout caused by various reasons.

A transversely isotropic model with a vertical symmetry axis (VTI medium) is the most commonly used anisotropic model for sedimentary basins, where the deviation

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from isotropy is usually attributed to some combination of fine layering and inherent anisotropy of shales. One of the first nonhyperbolic approximations for the P-wave reflection traveltimes in VTI media was proposed by Muir and Dellinger (1985) and further developed by Dellinger et al. (1993). Thomsen (1986) introduced a convenient parameterization of VTI media that was used by Tsvankin and Thomsen (1994) to describe nonhyperbolic reflection moveouts.

We begin with an overview of the weak-anisotropy approximation for P-wave velocities in VTI media and use it for analytic derivations throughout the paper. First, we consider a vertically heterogeneous anisotropic layer. For this model, we compare the three-parameter approximation for the P-wave traveltimes suggested by Tsvankin and Thomsen (1994) with the shifted hyperbola (Malovichko, 1978; Castle, 1988; de Bazelaire, 1988). Next, we examine P-wave moveout in VTI media above a curved reflector. We analyze the cumulative action of anisotropy, reflector dip, and reflector curvature, and develop an appropriate three-parameter representation for the reflection moveout. Finally, we consider models characterized by weak lateral heterogeneity and show that it can mimic the influence of transverse isotropy on nonhyperbolic moveout.

**WEAK ANISOTROPY APPROXIMATION FOR VTI MEDIA**

In transversely isotropic media, velocities of seismic waves depend on the direction of propagation measured from the symmetry axis. Thomsen (1986) introduced a notation for VTI media by replacing the elastic stiffness coefficients with the P- and S-wave velocities along the symmetry axis and three dimensionless anisotropic parameters. As shown by Tsvankin (1996), the P-wave seismic signatures in VTI media can be conveniently expressed in terms of Thomsen’s parameters $\epsilon$ and $\delta$. Deviations of these parameters from zero characterize the relative strength of anisotropy. For small values of these parameters, the weak-anisotropy approximation (Thomsen, 1986; Tsvankin and Thomsen, 1994) reduces to simple linearization.

The squared group velocity $V_g^2$ of P-waves in weakly anisotropic VTI media can be expressed as a function of the group angle $\psi$ measured from the vertical symmetry axis as follows:

$$V_g^2(\psi) = V_z^2 \left( 1 + 2\delta \sin^2 \psi \cos^2 \psi + 2\epsilon \sin^4 \psi \right),$$

(1)

where $V_z = V_g(0)$ is the P-wave vertical velocity, and $\delta$ and $\epsilon$ are Thomsen’s dimensionless anisotropic parameters, which are assumed to be small quantities:

$$|\epsilon| \ll 1, \quad |\delta| \ll 1.$$  

(2)

Both parameters are equal to zero in isotropic media.

Equation (1) is accurate up to the second-order terms in $\epsilon$ and $\delta$. We retain this level of accuracy throughout the paper. As follows from equation (1), the velocity $V_x$
corresponding to ray propagation in the horizontal direction is

\[ V_x^2 = V_g^2(\pi/2) = V_z^2 (1 + 2 \epsilon) . \]  

(3)

Equation (3) is actually exact, valid for any strength of anisotropy. Another important quantity is the normal-moveout (NMO) velocity, \( V_n \), that determines the small-offset \( P \)-wave reflection moveout in homogeneous VTI media above a horizontal reflector. Its exact expression is (Thomsen, 1986)

\[ V_n^2 = V_z^2 (1 + 2 \delta) . \]  

(4)

If \( \delta = 0 \) as, for example, in the ANNIE model proposed by Schoenberg et al. (1996), the normal-moveout velocity is equal to the vertical velocity.

It is convenient to rewrite equation (1) in the form

\[ V_g^2(\psi) = V_z^2 \left( 1 + 2 \delta \sin^2 \psi + 2 \eta \sin^4 \psi \right) , \]  

(5)

where

\[ \eta = \epsilon - \delta . \]  

(6)

Equation (6) is the weak-anisotropy approximation for the anellipticity coefficient \( \eta \) introduced by Alkhalifah and Tsvankin (1995). For the elliptic anisotropy, \( \epsilon = \delta \) and \( \eta = 0 \). To see why the group-velocity function becomes elliptic in this case, note that for small \( \delta \)

\[ \frac{1}{V_g^2(\psi)} \bigg|_{\eta=0} \approx \frac{1}{V_z^2 (1 + 2 \delta \sin^2 \psi)} \approx \frac{\cos^2 \psi}{V_z^2} + \frac{(1 - 2 \delta) \sin^2 \psi}{V_z^2} \approx \frac{\cos^2 \psi}{V_z^2} + \frac{\sin^2 \psi}{V_n^2} . \]  

(7)

Seismic data often indicate that \( \epsilon > \delta \), so the anellipticity coefficient \( \eta \) is usually positive.

An equivalent form of equation (1) can be obtained in terms of the three characteristic velocities \( V_z, V_x \), and \( V_n \):

\[ V^2_g(\psi) = V_z^2 \cos^2 \psi + \left( V_n^2 - V_x^2 \right) \sin^2 \psi \cos^2 \psi + V_x^2 \sin^2 \psi . \]  

(8)

From equation (8), in the linear approximation the anelliptic behavior of velocity is controlled by the difference between the normal moveout and horizontal velocities or, equivalently, by the difference between anisotropic coefficients \( \epsilon \) and \( \delta \).

We illustrate different types of the group velocities (wavefronts) in Figure 1. The wavefront, circular in the isotropic case (Figure 1a), becomes elliptical when \( \epsilon = \delta \neq 0 \) (Figure 1b). In the ANNIE model, the vertical and NMO velocities are equal (Figure 1c). If \( \epsilon > 0 \) and \( \delta < 0 \), the three characteristic velocities satisfy the inequality \( V_x > V_z > V_n \) (Figure 1d).
HORIZONTAL REFLECTOR BENEATH A HOMOGENEOUS VTI MEDIUM

To exemplify the use of weak anisotropy, let us consider the simplest model of a homogeneous VTI medium above a horizontal reflector. For an isotropic medium, the reflection traveltime curve is an exact hyperbola, as follows directly from the Pythagorean theorem (Figure 2)

\[ t^2(l) = \frac{4z^2 + l^2}{V_z^2} = t_0^2 + \frac{l^2}{V_z^2}, \quad (9) \]

where \( z \) denotes the depth of reflector, \( l \) is the offset, \( t_0 = t(0) \) is the zero-offset traveltime, and \( V_z \) is the isotropic velocity. For a homogeneous VTI medium, the velocity \( V_z \) in equation (9) is replaced by the angle-dependent group velocity \( V_g \). This replacement leads to the exact traveltimes if no approximation for the group velocity is used, since the ray trajectories in homogeneous VTI media remain straight, and the reflection point does not move. We can also obtain an approximate traveltime using the approximate velocity \( V_g \) defined by equations (1) or (5), where the ray angle \( \psi \) is given by

\[ \sin^2 \psi = \frac{l^2}{4z^2 + l^2}, \quad (10) \]
Substituting equation (10) into (5) and linearizing the expression

$$t^2(l) = \frac{4z^2 + l^2}{V_g^2(\psi)}$$

with respect to the anisotropic parameters $\delta$ and $\eta$, we arrive at the three-parameter nonhyperbolic approximation (Tsvankin and Thomsen 1994)

$$t^2(l) = t_0^2 + \frac{l^2}{V_n^2} - \frac{2\eta l^4}{V_n^2 (V_n^2 t_0^2 + l^2)}$$

(12)

where the normal-moveout velocity $V_n$ is defined by equation (4). At small offsets ($l \ll z$), the influence of the parameter $\eta$ is negligible, and the traveltime curve is nearly hyperbolic. At large offsets ($l \gg z$), the third term in equation (12) has a clear influence on the traveltime behavior. The Taylor series expansion of equation (12) in the vicinity of the vertical zero-offset ray has the form

$$t^2(l) = t_0^2 + \frac{l^2}{2V_n^2} - \frac{2\eta l^4}{4V_n^4 t_0^2} + \frac{2\eta l^6}{6V_n^6 t_0^4} - \ldots$$

(13)

When the offset $l$ approaches infinity, the traveltime approximately satisfies an intuitively reasonable relationship

$$\lim_{l \to \infty} t^2(l) = \frac{l^2}{V_x^2},$$

(14)

where the horizontal velocity $V_x$ is defined by equation (3). Approximation (12) is analogous, within the weak-anisotropy assumption, to the “skewed hyperbola” equation (Byun et al. 1989) which uses the three velocities $V_z$, $V_n$, and $V_x$ as the parameters of the approximation:

$$t^2(l) = t_0^2 + \frac{l^2}{V_n^2} - \frac{l^4}{V_n^4 (V_n^2 t_0^2 + l^2)}$$

(15)

The accuracy of equation (12), which usually lies within 1% error up to offsets twice as large as reflector depth, can be further improved at any finite offset by modifying the denominator of the third term (Alkhalifah and Tsvankin 1995; Grechka and Tsvankin 1998).

Muir and Dellinger (1985) suggested a different nonhyperbolic moveout approximation in the form

$$t^2(l) = t_0^2 + \frac{l^2}{V_n^2} - \frac{f (1 - f) l^4}{V_n^2 (V_n^2 t_0^2 + f l^2)}$$

(16)

where $f$ is the dimensionless parameter of anellipticity. At large offsets, equation (16) approaches

$$\lim_{l \to \infty} t^2(l) = f \frac{l^2}{V_n^2}.$$
Comparing equations (14) and (17), we can establish the correspondence

\[ f = \frac{V_n^2}{V_x^2} = \frac{1 + 2\delta}{1 + 2\epsilon} \approx 1 - 2\eta. \]  (18)

Taking this equality into account, we see that equation (16) is approximately equivalent to equation (12) in the sense that their difference has the order of \( \eta^2 \).

VERTICAL HETEROGENEITY

Vertical heterogeneity is another reason for nonhyperbolic moveout. We start this section by reviewing well-known results for isotropic media. Although these results can be interpreted in terms of an effective anisotropy, we show that it has different properties than those for the VTI model. We then extend the theory to vertically heterogeneous VTI media and perform a comparative analysis of various three-parameter nonhyperbolic approximations.

Vertically heterogeneous isotropic model

Nonhyperbolicity of reflection moveout in vertically heterogeneous isotropic media has been extensively studied using the Taylor series expansion in the powers of the offset (Bolshykh, 1956; Taner and Koehler, 1969; Al-Chalabi, 1973). The most important
property of vertically heterogeneous media is that the ray parameter
\[ p = \frac{\sin \psi(z)}{V_z(z)} \]
does not change along any given ray (Snell’s law). This fact leads to the explicit parametric relationships
\[ t(p) = 2 \int_0^z \frac{dz}{V_z(z)} \cos \psi(z) = \int_0^{t_z} \frac{dt_z}{\sqrt{1 - p^2 V_z^2(t_z)}}, \tag{19} \]
\[ l(p) = 2 \int_0^z dz \tan \psi(z) = \int_0^{t_z} \frac{p V_z^2(t_z) dt_z}{\sqrt{1 - p^2 V_z^2(t_z)}}, \tag{20} \]
where
\[ t_z = t(0) = 2 \int_0^z \frac{dz}{V_z(z)}. \tag{21} \]

Straightforward differentiation of parametric equations \(19\) and \(20\) yields the first four coefficients of the Taylor series expansion
\[ t^2(l) = a_0 + a_1 l^2 + a_2 l^4 + a_3 l^6 + \ldots \tag{22} \]
in the vicinity of the vertical zero-offset ray. Series \(22\) contains only even powers of the offset \(l\) because of the reciprocity principle: the pure-mode reflection traveltime is an even function of the offset. The Taylor series coefficients for the isotropic case are defined as follows:
\[ a_0 = t_z^2, \tag{23} \]
\[ a_1 = \frac{1}{V_{rms}^2}, \tag{24} \]
\[ a_2 = \frac{1 - S_2}{4 t_z^2 V_{rms}^4}, \tag{25} \]
\[ a_3 = \frac{2 S_2^2 - S_2 - S_3}{8 t_z^4 V_{rms}^6}, \tag{26} \]
where
\[ V_{rms}^2 = M_1, \tag{27} \]
\[ M_k = \frac{1}{t_z} \int_0^{t_z} V_z^{2k}(t) \, dt \quad (k = 1, 2, \ldots), \tag{28} \]
\[ S_k = \frac{M_k}{V_{rms}^{2k}} \quad (k = 2, 3, \ldots). \tag{29} \]

Equation \(24\) shows that, at small offsets, the reflection moveout has a hyperbolic form with the normal-moveout velocity \(V_n\) equal to the root-mean-square velocity.
At large offsets, however, the hyperbolic approximation is no longer accurate. Studying the Taylor series expansion (22), Malovichko (1978) introduced a three-parameter approximation for the reflection traveltime in vertically heterogeneous isotropic media. His equation has the form of a shifted hyperbola (Castle, 1988; de Bazelaire, 1988):

\[
t(l) = \left(1 - \frac{1}{S}\right) t_0 + \frac{1}{S} \sqrt{t_0^2 + S \frac{l^2}{V_n^2}}.
\]  

(30)

If we set the zero-offset traveltime \(t_0\) equal to the vertical traveltime \(t_z\), the velocity \(V_n\) equal to \(V_{rms}\), and the parameter of heterogeneity \(S\) equal to \(S^2\), equation (30) guarantees the correct coefficients \(a_0\), \(a_1\), and \(a_2\) in the Taylor series (22). Note that the parameter \(S^2\) is related to the variance \(\sigma^2\) of the squared velocity distribution, as follows:

\[
\sigma^2 = M_2 - V_{rms}^4 = V_{rms}^4 (S^2 - 1).
\]

(31)

According to equation (31), this parameter is always greater than unity (it equals 1 in homogeneous media). In many practical cases, the value of \(S^2\) lies between 1 and 2. We can roughly estimate the accuracy of approximation (30) at large offsets by comparing the fourth term of its Taylor series with the fourth term of the exact traveltime expansion (22). According to this estimate, the error of Malovichko’s approximation is

\[
\frac{\Delta t^2(l)}{t^2(0)} = \frac{1}{8} (S_3 - S^2) \left( \frac{l}{t_0 V_n} \right)^6.
\]

(32)

As follows from the definition of the parameters \(S_k\) [equations (29)] and the Cauchy-Schwartz inequality, the expression (32) is always nonnegative. This means that the shifted-hyperbola approximation tends to overestimate traveltimes at large offsets. As the offset approaches infinity, the limit of this approximation is

\[
\lim_{l \to \infty} t^2(l) = \frac{1}{S} \frac{l^2}{V_n^2}.
\]

(33)

Equation (33) indicates that the effective horizontal velocity for Malovichko’s approximation (the slope of the shifted hyperbola asymptote) differs from the normal-moveout velocity. One can interpret this difference as evidence of some effective depth-variant anisotropy. However, the anisotropy implied in equation (30) differs from the true anisotropy in a homogeneous transversely isotropic medium [see equation (1)]. To reveal this difference, let us substitute the effective values \(t(l) = \sqrt{4 z^2 + l^2/V_g(\psi)}\), \(t_0 = 2 z/V_z\), \(l = 2 z \tan \psi\), and \(S = V_z^2/V_n^2\) into equation (30). After eliminating the variables \(z\) and \(l\), the result takes the form

\[
\frac{1}{V_g(\psi)} = \frac{1}{V_z} \left\{ \cos \psi \left(1 - \frac{V_z^2}{V_n^2} \right) + \sqrt{\frac{V_z^2}{V_x^2} \sin^2 \psi + \frac{V_n^4}{V_x^4} \cos^2 \psi} \right\}.
\]

(34)

If the anisotropy is induced by vertical heterogeneity, \(V_x \geq V_n \geq V_z\). Those inequalities follow from the definitions of \(V_{rms}\), \(t_z\), \(S_2\), and the Cauchy-Schwartz inequality.
They reduce to equalities only when velocity is constant. Linearizing expression (34) with respect to Thomsen’s anisotropic parameters $\delta$ and $\epsilon$, we can transform it to the form analogous to that of equation (5):

$$V_g^2(\psi) = V_z^2 \left[ 1 + 2 \delta \sin^2 \psi + 2 \eta (1 - \cos \psi)^2 \right].$$

Figure 3 illustrates the difference between the VTI model and the effective anisotropy implied by the Malovichko approximation. The differences are noticeable in both the shapes of the effective wavefronts (Figure 3a) and the moveouts (Figure 3b).

![Wavefronts and Moveouts](image)

In deriving equation (35), we have assumed the correspondence

$$S = \frac{V_x^2}{V_n^2} = \frac{1 + 2 \epsilon}{1 + 2 \delta} \approx 1 + 2 \eta. \quad (36)$$

We could also have chosen the value of the parameter of heterogeneity $S$ that matches the coefficient $a_2$ given by equation (25) with the corresponding term in the Taylor series (13). Then, the value of $S$ is (Alkhalifah 1997)

$$S = 1 + 8 \eta. \quad (37)$$

The difference between equations (36) and (37) is an additional indicator of the fundamental difference between homogeneous VTI and vertically heterogeneous isotropic media. The three-parameter anisotropic approximation (12) can match the reflection moveout in the isotropic model up to the fourth-order term in the Taylor series expansion if the value of $\eta$ is chosen in accordance with equation (37). We can estimate the error of such an approximation with an equation analogous to (32):

$$\Delta t^2(l) = \frac{1}{8} (S_3 - 2 + 3 S_2 - 2 S_2^2) \left( \frac{l}{t_0 V_n} \right)^6. \quad (38)$$

The difference between the error estimates (32) and (38) is

$$\Delta t^2(l) = \frac{1}{8} (2 - S_2) (S_2 - 1) \left( \frac{l}{t_0 V_n} \right)^6. \quad (39)$$
For usual values of $S_2$, which range from 1 to 2, the expression (39) is positive. This means that the anisotropic approximation (12) overestimates traveltimes in the isotropic heterogeneous model even more than does the shifted hyperbola (30) shown in Figure 3b. Below, we examine which of the two approximations is more suitable when the model includes both vertical heterogeneity and anisotropy.

**Vertically heterogeneous VTI model**

In a model that includes vertical heterogeneity and anisotropy, both factors influence bending of the rays. The weak anisotropy approximation, however, allows us to neglect the effect of anisotropy on ray trajectories and consider its influence on traveltimes only. This assumption is analogous to the linearization, conventionally done for tomographic inversion. Its application to weak anisotropy has been discussed by [Grechka and McMechan 1996]. According to the linearization assumption, we can retain isotropic equation (20) describing the ray trajectories and rewrite equation (19) in the form

$$t(p) = 2 \int_0^z \frac{dz}{V_g(z, \psi(z)) \cos \psi(z)},$$

where $V_g$ is the anisotropic group velocity, which varies both with the depth $z$ and with the ray angle $\psi$ and has the expression (1). Differentiation of the parametric traveltime equations (40) and (20) and linearization with respect to Thomsen’s anisotropic parameters shows that the general form of equations (23)–(26) remains valid if we replace the definitions of the root-mean-square velocity $V_{rms}$ and the parameters $M_k$ by

$$V_{rms}^2 = \frac{1}{t_z} \int_0^{t_z} V_z^2(t) \left[ 1 - \delta(t) \right] dt,$$

$$M_k = \frac{1}{t_z} \int_0^{t_z} V_z^{2k}(t) \left[ 1 - \delta(t) \right]^{2k} \left[ 1 + 8 \eta(t) \right] dt \quad (k = 2, 3, \ldots).$$

In homogeneous media, expressions (41) and (42) transform series (22) with coefficients (23)–(26) into the form equivalent to series (13). Two important conclusions follow from equations (41) and (42). First, if the mean value of the anisotropic coefficient $\delta$ is less than zero, the presence of anisotropy can reduce the difference between the effective root-mean-square velocity and the effective vertical velocity $\hat{V}_z = z/t_z$. In this case, the influence of anisotropy and heterogeneity partially cancel each other, and the moveout curve may behave at small offsets as if the medium were homogeneous and isotropic. This behavior has been noticed by [Larner and Cohen 1993]. On the other hand, if the anellipticity coefficient $\eta$ is positive and different from zero, it can significantly increase the values of the heterogeneity parameters $S_k$ defined by equations (29). Then, the nonhyperbolicity of reflection moveouts at large offsets is stronger than that in isotropic media.

To exemplify the general theory, let us consider a simple analytic model with constant anisotropic parameters and the vertical velocity linearly increasing with
depth according to the equation
\[ V_z(z) = V_z(0) (1 + \beta z) = V_z(0) e^{\kappa(z)} , \tag{43} \]
where \( \kappa \) is the logarithm of the velocity change. In this case, the analytic expression for the RMS velocity \( V_{rms}^2 \) is found from equation (41) to be
\[ V_{rms}^2 = V_z^2(0) (1 + 2 \delta) \frac{e^{2\kappa} - 1}{2\kappa} , \tag{44} \]
while the mean vertical velocity is
\[ \bar{V}_z = \frac{z}{t_z} = V_z(0) \frac{e^{\kappa} - 1}{\kappa} , \tag{45} \]
where \( \kappa = \kappa(z) \) is evaluated at the reflector depth. Comparing equations (44) and (45), we can see that the squared RMS velocity \( V_{rms}^2 \) equals to the squared mean velocity \( \bar{V}_z^2 \) if
\[ 1 + 2 \delta = \frac{2 (e^{\kappa} - 1)}{\kappa (e^{\kappa} + 1)} . \tag{46} \]
For small \( \kappa \), the estimate of \( \delta \) from equation (46) is
\[ \delta \approx -\frac{\kappa^2}{24} . \tag{47} \]
For example, if the vertical velocity near the reflector is twice that at the surface (i.e., \( \kappa = \ln 2 \approx 0.69 \)), having the anisotropic parameter \( \delta \) as small as \(-0.02\) is sufficient to cancel out the influence of heterogeneity on the normal-moveout velocity. The values of parameters \( S_2 \) and \( S_3 \), found from equations (29), (41) and (42), are
\[ S_2 = (1 + 8 \eta) \kappa \frac{e^{2\kappa} + 1}{e^{2\kappa} - 1} , \tag{48} \]
\[ S_3 = \frac{4}{3} (1 + 8 \eta) \kappa^2 \frac{e^{4\kappa} + e^{2\kappa} + 1}{(e^{2\kappa} - 1)^2} . \tag{49} \]
Substituting equations (48) and (49) into the estimates (32) and (38) and linearizing them both in \( \eta \) and in \( \kappa \), we find that the error of anisotropic traveltime approximation (12) in the linear velocity model is
\[ \Delta t^2(l) t^2(0) = -\frac{\kappa^2 (1 - 8 \eta)}{12} \left( \frac{l}{t_0 V_n} \right)^6 , \tag{50} \]
while the error of the shifted-hyperbola approximation (30) is
\[ \Delta t^2(l) t^2(0) = \left( \frac{\kappa^2 (1 - 8 \eta)}{24} - \eta \right) \left( \frac{l}{t_0 V_n} \right)^6 . \tag{51} \]
Comparing equations (50) and (51), we conclude that if the medium is elliptically anisotropic (\( \eta = 0 \)), the shifted hyperbola can be twice as accurate as the anisotropic
equation (assuming the optimal choice of parameters). The accuracy of the latter, however, increases when the anellipticity coefficient \( \eta \) grows and becomes higher than that of the shifted hyperbola if \( \eta \) satisfies the approximate inequality

\[
\eta \geq \frac{\kappa^2}{8 (1 + \kappa^2)} .
\]

(52)

For instance, if \( \kappa = \ln 2 \), inequality (52) yields \( \eta \geq 0.03 \), a quite small value.

**CURVILINEAR REFLECTOR**

Reflector curvature can also cause nonhyperbolic reflection moveout. In isotropic media, local dip of the reflector influences the normal-moveout velocity, while reflector curvature introduces nonhyperbolic moveout. When overlaying layer is also anisotropic, both hyperbolic and nonhyperbolic moveouts for reflections from curved reflectors also become functions of the anisotropic parameters.

**Curved reflector beneath isotropic medium**

If the reflector has the shape of a dipping plane beneath a homogeneous isotropic medium, the reflection moveout in the dip direction is a hyperbola \([\text{Levin}, 1971]\)

\[
t^2(l) = t_0^2 + \frac{l^2}{V_n^2} .
\]

(53)

Here

\[
t_0 = \frac{2L}{V_z} ,
\]

(54)

\[
V_n = \frac{V_z}{\cos \alpha} ,
\]

(55)

\( L \) is the length of the zero-offset ray, and \( \alpha \) is the reflector dip. Formula (53) is inaccurate if the reflector is both dipping and curved. The Taylor series expansion for moveout in this case has the form of equation (22), with coefficients \([\text{Fomel}, 1994]\)

\[
a_2 = \frac{\cos^2 \alpha \sin^2 \alpha G}{4V_z^2 L^2} ,
\]

(56)

\[
a_3 = -\frac{\cos^2 \alpha \sin^2 \alpha G^2}{16V_z^2 L^4} \left( \cos 2\alpha + \sin 2\alpha \frac{G K_3}{K_2 L} \right) ,
\]

(57)

where

\[
G = \frac{K_2 L}{1 + K_2 L} ,
\]

(58)

\( K_2 \) is the reflector curvature \([\text{defined by equation (61)}]\) at the reflection point of the zero-offset ray, and \( K_3 \) is the third-order curvature \([\text{equation (62)}]\). If the reflector
has an explicit representation $z = z(x)$, then the parameters in equations (56) and (57) are

$$\tan \alpha = \frac{dz}{dx}, \tag{59}$$
$$L = \frac{z}{\cos \alpha}, \tag{60}$$
$$K_2 = \frac{d^2z}{dx^2} \cos^3 \alpha, \tag{61}$$
$$K_3 = \frac{d^3z}{dx^3} \cos^4 \alpha - 3 K_2^2 \tan \alpha. \tag{62}$$

Keeping only three terms in the Taylor series leads to the approximation

$$t^2(l) = t_0^2 + \frac{l^2}{V_n^2} + \frac{G l^4 \tan^2 \alpha}{V_n^2 (V_n^2 t_0^2 + G l^2)}, \tag{63}$$

where we included the denominator in the third term to ensure that the traveltime behavior at large offsets satisfies the obvious limit

$$\lim_{l \to \infty} t^2(l) = \frac{l^2}{V_z^2}. \tag{64}$$

As indicated by equation (61), the sign of the curvature $K_2$ is positive if the reflector is locally convex (i.e., an anticline-type). The sign of $K_2$ is negative for concave, syncline-type reflectors. Therefore, the coefficient $G$ expressed by equation (58) and, likewise, the nonhyperbolic term in (63) can take both positive and negative values. This means that only for concave reflectors in homogeneous media do nonhyperbolic moveouts resemble those in VTI and vertically heterogeneous media. Convex surfaces produce nonhyperbolic moveout with the opposite sign. Clearly, equation (63) is not accurate for strong negative curvatures $K_2 \approx -1/L$, which cause focusing of the reflected rays and triplications of the reflection traveltimes.

In order to evaluate the accuracy of approximation (63), we can compare it with the exact expression for a point diffractor, which is formally a convex reflector with an infinite curvature. The exact expression for normal moveout in the present notation is

$$t(l) = \frac{\sqrt{z^2 + (z \tan \alpha - l/2)^2} + \sqrt{z^2 + (z \tan \alpha + l/2)^2}}{V_z}, \tag{65}$$

where $z$ is the depth of the diffractor, and $\alpha$ is the angle from vertical of the zero-offset ray. Figure 4 shows the relative error of approximation (63) as a function of the ray angle for offset $l$ twice the diffractor depth $z$. The maximum error of about 1% occurs at $\alpha \approx 50^\circ$. We can expect equation (63) to be even more accurate for reflectors with smaller curvatures.
Curved reflector beneath homogeneous VTI medium

For a dipping curved reflector in a homogeneous VTI medium, the ray trajectories of the incident and reflected waves are straight, but the location of the reflection point is no longer controlled by the isotropic laws. To obtain analytic expressions in this model, we use the theorem that connects the derivatives of the common-midpoint traveltime with the derivatives of the one-way traveltimes for an imaginary wave originating at the reflection point of the zero-offset ray. This theorem, introduced for the second-order derivatives by Chernjak and Gritsenko (1979), is usually called the normal incidence point (NIP) theorem (Hubral and Krey, 1980; Hubral, 1983). Although the original proof did not address anisotropy, it is applicable to anisotropic media because it is based on the fundamental Fermat’s principle. The “normal incidence” point in anisotropic media is the point of incidence for the zero-offset ray (which is, in general, not normal to the reflector). In Appendix A, we review the NIP theorem, as well as its extension to the high-order traveltime derivatives (Fomel, 1994).

Two important equations derived in Appendix A are:

\[
\frac{\partial^2 t}{\partial l^2} \bigg|_{l=0} = \frac{1}{2} \frac{\partial^2 T}{\partial y^2}, \\
\frac{\partial^4 t}{\partial l^4} \bigg|_{l=0} = \frac{1}{8} \frac{\partial^4 T}{\partial y^4} - \frac{3}{8} \left( \frac{\partial^2 T}{\partial x^2} \right)^{-1} \left( \frac{\partial^3 T}{\partial y^2 \partial x} \right)^2,
\]

where \(T(x, y)\) is the one-way traveltime of the direct wave propagating from the reflection point \(x\) to the point \(y\) at the surface \(z = 0\). All derivatives in equations (66) and (67) are evaluated at the zero-offset ray. Both equations are based solely on Fermat’s principle and, therefore, remain valid in any type of media for reflectors of an arbitrary shape, assuming that the traveltimes possess the required order of

\[\text{Relative Error} \]

\[\text{angle} / \text{Minus} 0.005 \]

\[0.005 \]

\[0.010 \]

\[20 40 60 80\]

\[\text{Relative Error} \]

\[\text{angle} / \text{Minus} 0.005 \]

\[0.005 \]

\[0.010 \]

\[\text{Relative Error} \]

Figure 4: Relative error \(e\) of the nonhyperbolic moveout approximation (63) for a point diffractor. The error corresponds to offset \(l\) twice the diffractor depth \(z\) and is plotted against the angle from vertical \(\alpha\) of the zero-offset ray.
smoothness. It is especially convenient to use equations (66) and (67) in homogeneous media, where the direct traveltime $T$ can be expressed explicitly.

To apply equations (66) and (67) in VTI media, we need to start with tracing the zero-offset ray. According to Fermat’s principle, the ray trajectory must correspond to an extremum of the traveltime. For the zero-offset ray, this simply means that the one-way traveltime $T$ satisfies the equation

$$\frac{\partial T}{\partial x} = 0,$$  
(68)

where

$$T(x, y) = \frac{\sqrt{z^2(x) + (x - y)^2}}{V_g(\psi(x, y))}. $$  
(69)

Here, the function $z(x)$ describes the reflector shape, and $\psi$ is the ray angle given by the trigonometric relationship (Figure 5)

$$\cos \psi(x, y) = \frac{z(x)}{\sqrt{z^2(x) + (x - y)^2}}.$$

Substituting approximate equation (5) for the group velocity $V_g$ into equation (69) and linearizing it with respect to the anisotropic parameters $\delta$ and $\eta$, we can solve equation (68) for $y$, obtaining

$$y = x + z \tan \alpha \left(1 + 2\delta + 4\eta \sin^2 \alpha\right) $$

or, in terms of $\psi$,

$$\tan \psi = \tan \alpha \left(1 + 2\delta + 4\eta \sin^2 \alpha\right),$$

(72)

where $\alpha$ is the local dip of the reflector at the reflection point $x$. Equation (72) shows that, in VTI media, the angle $\psi$ of the zero-offset ray differs from the reflector dip $\alpha$ (Figure 5). As one might expect, the relative difference is approximately linear in Thomsen anisotropic parameters.

Now we can apply equation (66) to evaluate the second term of the Taylor series expansion (22) for a curved reflector. The linearization in anisotropic parameters leads to the expression

$$a_1 = \frac{1}{V_n^2} = \frac{\cos^2 \alpha}{V_z^2 \left(1 + 2\delta \left(1 + \sin^2 \alpha\right) + 6\eta \sin^2 \alpha \left(1 + \cos^2 \alpha\right)\right)},$$

(73)

which is equivalent to that derived by Tsvankin (1995). As in isotropic media, the normal-moveout velocity does not depend on the reflector curvature. Its dip dependence, however, is an important indicator of anisotropy, especially in areas of conflicting dips (Alkhalifah and Tsvankin, 1995).

Finally, using equation (67), we determine the third coefficient of the Taylor series. After linearization in anisotropic parameters and lengthy algebra, the result takes the form

$$a_2 = \frac{A}{V_n^4 t_0^2},$$

(74)
Figure 5: Zero-offset reflection from a curved reflector beneath a VTI medium (a scheme). Note that the ray angle $\psi$ is not equal to the local reflector dip $\alpha$. 

$\alpha$ $\psi$ $z$ $x$ $y$ $\alpha$ 

aniso/XFig nmoray
where
\[
A = G \tan^2 \alpha + 2 \delta G \sin^2 \alpha (2 + \tan^2 \alpha - G) - 2 \eta (1 - 4 \sin^2 \alpha) +
+ 4 \eta G \sin^2 \alpha \left(6 \cos^2 \alpha + \sin^2 \alpha (\tan^2 \alpha - 3 G)\right),
\] (75)
and the coefficient \(G\) is defined by equation (58). For zero curvature (a plane reflector) \(G = 0\), and the only term remaining in equation (75) is
\[
A = -2 \eta (1 - 4 \sin^2 \alpha).
\] (76)
If the reflector is curved, we can rewrite the isotropic equation (63) in the form
\[
t^2(l) = t_0^2 + \frac{l^2}{V_n^2} + \frac{A l^4}{V_n^2 (V_n^2 t_0^2 + G l^2)},
\] (77)
where the normal-moveout velocity \(V_n\) and the quantity \(A\) are given by equations (73) and (75), respectively. Equation (77) approximates the nonhyperbolic moveout in homogeneous VTI media above a curved reflector. For small curvature, the accuracy of this equation at finite offsets can be increased by modifying the denominator in the quartic term similarly to that done by Grechka and Tsvankin (1998) for VTI media.

**ANISOTROPY VERSUS LATERAL HETEROGENEITY**

The nonhyperbolic moveout in homogeneous VTI media with one horizontal reflector is similar to that caused by lateral heterogeneity in isotropic models. In this section, we discuss this similarity following the results of Grechka (1998).

The angle dependence of the group velocity in equations (1) and (5) is characterized by small anisotropic coefficients. Therefore, we can assume that an analogous influence of lateral heterogeneity might be caused by small velocity perturbations. (Large lateral velocity changes can cause behavior too complicated for analytic description.) An appropriate model is a plane laterally heterogeneous layer with the velocity
\[
V(y) = V_0 [1 + c(y)],
\] (78)
where \(|c(y)| \ll 1\) is a dimensionless function. The velocity \(V(y)\) given by equation (78) has the generic perturbation form that allows us to use the tomographic linearization assumption. That is, we neglect the ray bending caused by the small velocity perturbation \(c\) and compute the perturbation of traveltimes along straight rays in the constant-velocity background. Thus, we can rewrite equation (9) as
\[
t(l) = \frac{\sqrt{4 z^2 + l^2}}{l} \int_{y-l/2}^{y+l/2} \frac{d\xi}{V_z(\xi)},
\] (79)
where \(y\) is the midpoint location and the integration limits correspond to the source and receiver locations. For simplicity and without loss of generality, we can set \(y\) to
zero. Linearizing equation (79) with respect to the small perturbation \( c(y) \), we get

\[
t(l) = \frac{\sqrt{4 \, z^2 + l^2}}{V_0} \left[ 1 - \frac{1}{l} \int_{-l/2}^{l/2} c(\xi) \, d\xi \right].
\]  

(80)

It is clear from equation (80) that lateral heterogeneity can cause many different types of the nonhyperbolic moveout. In particular, comparing equations (80) and (11), we conclude that a pseudo-anisotropic behavior of traveltimes is produced by lateral heterogeneity in the form

\[
c(l) = \frac{d}{dl} \left[ \frac{l^3 (l^2 \epsilon + 4 \, z^2 \delta)}{(l^2 + 4 \, z^2)^2} \right]
\]  

(81)

or, in the linear approximation,

\[
c(l) = \frac{4 \, \delta \, t_0^2 V_n^2 l^2 (3 \, t_n^2 V_n^2 - l^2) + \epsilon \, l^4 (5 \, t_n^2 V_n^2 + l^2)}{16 \, (t_n^2 V_n^2 + l^2)^3},
\]  

(82)

where \( \delta \) and \( \epsilon \) should be considered now as parameters, describing the isotropic laterally heterogeneous velocity field. Equation (82) indicates that the velocity heterogeneity \( c(y) \) that reproduces moveout (12) in a homogeneous VTI medium, is a symmetric function of the offset \( l \). This is not surprising because the velocity function (1), corresponding to vertical transverse isotropy, is symmetric as well.

CONCLUSIONS

Nonhyperbolic reflection moveout of \( P \)-waves is sometimes considered as an important indicator of anisotropy. Its correct interpretation, however, is impossible without taking other factors into account. In this paper, we have considered three other important factors: vertical heterogeneity, curvature of the reflector, and lateral heterogeneity. Each of them can have an influence on nonhyperbolic behavior of the reflection moveout comparable to that of anisotropy. In particular, vertical heterogeneity produces a depth-variant anisotropic pattern that differs from that in VTI media. For isotropic media, this pattern is reasonably well approximated by the shifted hyperbola. In a vertically heterogeneous VTI medium, the parameters of anisotropy should be replaced with their effective values. For a curved reflector in a homogeneous VTI medium, we have developed an approximation based on the Taylor series expansion of the traveltime with both the reflector curvature and the anisotropic parameters entering the nonhyperbolic term. Lateral heterogeneity can effectively mimic the influence of virtually any anisotropy.

The theoretical results of this paper are directly applicable to modeling of the nonhyperbolic moveout. The general formulas connecting the derivatives of reflection traveltimes with those of direct waves are particularly attractive in this context. For
smooth velocity models, these formulas reduce the problem of tracing a family of reflected rays to tracing only one zero-offset ray. Practical estimation and inversion of nonhyperbolic moveout is a different and more difficult problem than is the forward one. Given that a variety of reasons might cause similar nonhyperbolic moveout of $P$-waves, its inversion will be nonunique. Nevertheless, the theoretical guidelines provided by the analytical theory are helpful for the correct formulation of the inverse problems. They explicitly show us which medium parameters we may hope to extract from the kinematics of long-spread $P$-wave reflection data.

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APPENDIX A

NORMAL MOVEOUT BEYOND THE NIP THEOREM

In this Appendix, we derive equations that relate traveltime derivatives of the reflected wave, evaluated at the zero offset point, and traveltime derivatives of the direct wave, evaluated in the vicinity of the zero-offset ray. Such a relationship for second-order derivatives is known as the NIP (normal incidence point) theorem (Chernjak and Gritsenko [1979]; Hubral and Krey [1980]; Hubral [1983]). Its extension to high-order derivatives is described by Fomel (1994).

Reflection traveltime in any type of model can be considered as a function of the source and receiver locations $s$ and $r$ and the location of the reflection point $x$, as follows:

$$t(y, h) = F(y, h, x(y, h)),$$  \hspace{1cm} (A-1)

where $y$ is the midpoint ($y = \frac{s + r}{2}$), $h$ is the half-offset ($h = \frac{r - s}{2}$), and the function $F$ has a natural decomposition into two parts corresponding to the incident and reflected rays:

$$F(y, h, x) = T(y - h, x) + T(y + h, x),$$  \hspace{1cm} (A-2)

where $T$ is the traveltine of the direct wave. Clearly, at the zero-offset point,

$$t(y, 0) = 2T(y, x),$$  \hspace{1cm} (A-3)

where $x = x(y, 0)$ corresponds to the reflection point of the zero-offset ray.
Differentiating equation (A-1) with respect to the half-offset $h$ and applying the chain rule, we obtain
\[ \frac{\partial t}{\partial h} = \frac{\partial F}{\partial h} + \frac{\partial F}{\partial x} \frac{\partial x}{\partial h}. \] (A-4)

According to Fermat’s principle, one of the fundamental principles of ray theory, the ray trajectory of the reflected wave corresponds to an extremum value of the traveltime. Parameterizing the trajectory in terms of the reflection point location $x$ and assuming that $F$ is a smooth function of $x$, we can write Fermat’s principle in the form
\[ \frac{\partial F}{\partial x} = 0. \] (A-5)

Equation (A-5) must be satisfied for any values of $x$ and $h$. Substituting this equation into equation (A-4) leads to the equation
\[ \frac{\partial t}{\partial h} = \frac{\partial F}{\partial h}. \] (A-6)

Differentiating (A-6) again with respect to $h$, we arrive at the equation
\[ \frac{\partial^2 t}{\partial h^2} = \frac{\partial^2 F}{\partial h^2} + \frac{\partial^2 F}{\partial h \partial x} \frac{\partial x}{\partial h}. \] (A-7)

Interchanging the source and receiver locations doesn’t change the reflection point position (the principle of reciprocity). Therefore, $x$ is an even function of the offset $h$, and we can simplify equation (A-7), as follows:
\[ \left. \frac{\partial^2 t}{\partial h^2} \right|_{h=0} = \left. \frac{\partial^2 F}{\partial h^2} \right|_{h=0}. \] (A-8)

Substituting the expression for the function $F$ (A-2) into (A-8) leads to the equation
\[ \left. \frac{\partial^2 t}{\partial h^2} \right|_{h=0} = 2 \left. \frac{\partial^2 T}{\partial y^2} \right|_{h=0}, \] (A-9)

which is the mathematical formulation of the NIP theorem. It proves that the second-order derivative of the reflection traveltime with respect to the offset is equal, at zero offset, to the second derivative of the direct wave traveltime for the wave propagating from the incidence point of the zero-offset ray. One immediate conclusion from the NIP theorem is that the short-spread normal moveout velocity, connected with the derivative in the left-hand-side of equation (A-9) can depend on the reflector dip but doesn’t depend on the curvature of the reflector. Our derivation up to this point has followed the derivation suggested by Chernjak and Gritsenko (1979).

Differentiating equation (A-7) twice with respect to $h$ evaluates, with the help of the chain rule, the fourth-order derivative, as follows:
\[ \frac{\partial^4 t}{\partial h^4} = \frac{\partial^4 F}{\partial h^4} + 3 \frac{\partial^4 F}{\partial h^3 \partial x} \frac{\partial x}{\partial h} + 3 \frac{\partial^4 F}{\partial h^2 \partial x^2} \left( \frac{\partial x}{\partial h} \right)^2 + 3 \frac{\partial^4 F}{\partial h \partial x^3} \left( \frac{\partial x}{\partial h} \right)^3 + \]
Again, we can apply the principle of reciprocity to eliminate the odd-order derivatives of $x$ in equation (A-10) at the zero offset. The resultant expression has the form
\[ \frac{\partial^4 t}{\partial h^4} \bigg|_{h=0} = \left( \frac{\partial^4 F}{\partial h^4} + 3 \frac{\partial^3 F}{\partial h^2 \partial x} \frac{\partial x}{\partial h^2} \right) \bigg|_{h=0}. \] (A-11)

In order to determine the unknown second derivative of the reflection point location $\frac{\partial^2 x}{\partial h^2}$, we differentiate Fermat’s equation (A-5) twice, obtaining
\[ \frac{\partial^3 F}{\partial h \partial x} + 2 \frac{\partial^3 F}{\partial h \partial x} \frac{\partial x}{\partial h} + \frac{\partial^3 F}{\partial x} \left( \frac{\partial x}{\partial h} \right)^2 + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 x}{\partial h^2} = 0. \] (A-12)

Simplifying this equation at zero offset, we can solve it for the second derivative of $x$. The solution has the form
\[ \frac{\partial^2 x}{\partial h^2} \bigg|_{h=0} = -\left( \left( \frac{\partial^2 F}{\partial x^2} \right)^{-1} \frac{\partial^3 F}{\partial h \partial x} \right) \bigg|_{h=0}. \] (A-13)

Here we neglect the case of $\frac{\partial^2 F}{\partial x^2} = 0$, which corresponds to a focusing of the reflected rays at the surface. Finally, substituting expression (A-13) into (A-11) and recalling the definition of the $F$ function from (A-2), we obtain the equation
\[ \frac{\partial^4 t}{\partial h^4} \bigg|_{h=0} = 2 \frac{\partial^4 T}{\partial y^4} - \frac{\partial^3 T}{\partial x^2} \left( \frac{\partial^2 T}{\partial y^2 \partial x} \right)^2, \] (A-14)

which is the same as equation (67) in the main text. Higher-order derivatives can be expressed in an analogous way with a set of recursive algebraic functions (Fomel, 1994).

In the derivation of equations (A-9) and (A-14), we have used Fermat’s principle, the principle of reciprocity, and the rules of calculus. Both these equations remain valid in anisotropic media as well as in heterogeneous media, providing that the traveltime function is smooth and that focusing of the reflected rays doesn’t occur at the surface of observation.
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Time-shift imaging condition in seismic migration

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**ABSTRACT**

Seismic imaging based on single-scattering approximation is based on analysis of the match between the source and receiver wavefields at every image location. Wavefields at depth are functions of space and time and are reconstructed from surface data either by integral methods (Kirchhoff migration) or by differential methods (reverse-time or wavefield extrapolation migration). Different methods can be used to analyze wavefield matching, of which cross-correlation is a popular option. Implementation of a simple imaging condition requires time cross-correlation of source and receiver wavefields, followed by extraction of the zero time lag. A generalized imaging condition operates by cross-correlation in both space and time, followed by image extraction at zero time lag. Images at different spatial cross-correlation lags are indicators of imaging accuracy and are also used for image angle-decomposition.

In this paper, we introduce an alternative prestack imaging condition in which we preserve multiple lags of the time cross-correlation. Prestack images are described as functions of time-shifts as opposed to space-shifts between source and receiver wavefields. This imaging condition is applicable to migration by Kirchhoff, wavefield extrapolation or reverse-time techniques. The transformation allows construction of common-image gathers presented as function of either time-shift or reflection angle at every location in space. Inaccurate migration velocity is revealed by angle-domain common-image gathers with non-flat events. Computational experiments using a synthetic dataset from a complex salt model demonstrate the main features of the method.

**INTRODUCTION**

A key challenge for imaging in complex areas is accurate determination of a velocity model in the area under investigation. Migration velocity analysis is based on the principle that image accuracy indicators are optimized when data are correctly imaged. A common procedure for velocity analysis is to examine the alignment of images created with multi-offset data. An optimal choice of image analysis can be done in the angle domain which is free of some complicated artifacts present in surface offset gathers in complex areas [Stolk and Symes 2004].

Migration velocity analysis after migration by wavefield extrapolation requires image decomposition in scattering angles relative to reflector normals. Several methods
have been proposed for such decompositions (de Bruin et al., 1990; Prucha et al., 1999; Mosher and Foster, 2000; Rickett and Sava, 2002; Xie and Wu, 2002; Sava and Fomel, 2003; Soubra, 2003; Fomel, 2004; Biondi and Symes, 2004). These procedures require decomposition of extrapolated wavefields in variables that are related to the reflection angle.

A key component of such image decompositions is the imaging condition. A careful implementation of the imaging condition preserves all information necessary to decompose images in their angle-dependent components. The challenge is efficient and reliable construction of these angle-dependent images for velocity or amplitude analysis.

In migration with wavefield extrapolation, a prestack imaging condition based on spatial shifts of the source and receiver wavefields allows for angle-decomposition (Rickett and Sava, 2002; Sava and Fomel, 2005). Such formed angle-gathers describe reflectivity as a function of reflection angles and are powerful tools for migration velocity analysis (MVA) or amplitude versus angle analysis (AVA). However, due to the large expense of space-time cross-correlations, especially in three dimensions, this imaging methodology is not used routinely in data processing.

This paper presents a different form of imaging condition. The key idea of this new method is to use time-shifts instead of space-shifts between wavefields computed from sources and receivers. Similarly to the space-shift imaging condition, an image is built by space-time cross-correlations of subsurface wavefields, and multiple lags of the time cross-correlation are preserved in the image. Time-shifts have physical meaning that can be related directly to reflection geometry, similarly to the procedure used for space-shifts. Furthermore, time-shift imaging is cheaper to apply than space-shift imaging, and thus it might alleviate some of the difficulties posed by costly cross-correlations in 3D space-shift imaging condition.

The idea of a time-shift imaging condition is related to the idea of depth focusing analysis (Faye and Jeannot, 1986; MacKay and Abma, 1992, 1993; Nemeth, 1995, 1996). The main novelty of our approach is that we employ time-shifting to construct angle-domain gathers for prestack depth imaging.

The time-shift imaging concept is applicable to Kirchhoff migration, migration by wavefield extrapolation, or reverse-time migration. We present a theoretical analysis of this new imaging condition, followed by a physical interpretation leading to angle-decomposition. Finally, we illustrate the method with images of the complex Sigsbee 2A dataset (Paffenholz et al., 2002).

**IMAGING CONDITION IN WAVE-EQUATION IMAGING**

A traditional imaging condition for shot-record migration, often referred-to as $UD^*$ imaging condition (Claerbout, 1985), consists of time cross-correlation at every image location between the source and receiver wavefields, followed by image extraction at
zero time:
\[ U(m, t) = U_r(m, t) * U_s(m, t) \],
\[ R(m) = U(m, t = 0) \],
where the symbol \( * \) denotes cross-correlation in time. Here, \( m = [m_x, m_y, m_z] \) is a vector describing the locations of image points, \( U_s(m, t) \) and \( U_r(m, t) \) are source and receiver wavefields respectively, and \( R(m) \) denotes a migrated image. A final image is obtained by summation over shots.

**Space-shift imaging condition**

A generalized prestack imaging condition \cite{Sava2005} estimates image reflectivity using cross-correlation in space and time, followed by image extraction at zero time:
\[ U(m, h, t) = U_r(m + h, t) * U_s(m - h, t) \],
\[ R(m, h) = U(m, h, t = 0) \].
Here, \( h = [h_x, h_y, h_z] \) is a vector describing the space-shift between the source and receiver wavefields prior to imaging. Special cases of this imaging condition are horizontal space-shift \cite{Rickett2002} and vertical space-shift \cite{Biondi2004}.

For computational reasons, this imaging condition is usually implemented in the Fourier domain using the expression
\[ R(m, h) = \sum_\omega U_r(m + h, \omega) U_s^*(m - h, \omega) e^{2\pi i \omega \tau} \].
The \( * \) sign represents a complex conjugate applied on the receiver wavefield \( U_s \) in the Fourier domain.

**Time-shift imaging condition**

Another possible imaging condition, advocated in this paper, involves shifting of the source and receiver wavefields in time, as opposed to space, followed by image extraction at zero time:
\[ U(m, t, \tau) = U_r(m, t + \tau) * U_s(m, t - \tau) \],
\[ R(m, \tau) = U(m, \tau, t = 0) \].
Here, \( \tau \) is a scalar describing the time-shift between the source and receiver wavefields prior to imaging. This imaging condition can be implemented in the Fourier domain using the expression
\[ R(m, \tau) = \sum_\omega U_r(m, \omega) U_s^*(m, \omega) e^{2\pi i \omega \tau} \].
which simply involves a phase-shift applied to the wavefields prior to summation over frequency $\omega$ for imaging at zero time.

**Space-shift and time-shift imaging condition**

To be even more general, we can formulate an imaging condition involving both space-shift and time-shift, followed by image extraction at zero time:

$$U(m, h, t) = U_r(m + h, t + \tau) * U_s(m - h, t - \tau), \quad (9)$$
$$R(m, h, \tau) = U(m, h, \tau, t = 0). \quad (10)$$

However, the cost involved in this transformation is large, so this general form does not have immediate practical value. Imaging conditions described by equations (3)-(4) and (6)-(7) are special cases of equations (9)-(10) for $h = 0$ and $\tau = 0$, respectively.

**ANGLE TRANSFORMATION IN WAVE-EQUATION IMAGING**

Using the definitions introduced in the preceding section, we can make the standard notations for source and receiver coordinates: $s = m - h$ and $r = m + h$. The traveltime from a source to a receiver is a function of all spatial coordinates of the seismic experiment $t = t(m, h)$. Differentiating $t$ with respect to all components of the vectors $m$ and $h$, and using the standard notations $p_\alpha = \nabla_\alpha t$, where $\alpha = \{m, h, s, r\}$, we can write:

$$p_m = p_r + p_s, \quad (11)$$
$$p_h = p_r - p_s. \quad (12)$$

From equations (11)-(12), we can write

$$2p_s = p_m - p_h, \quad (13)$$
$$2p_r = p_m + p_h. \quad (14)$$

By analyzing the geometric relations of various vectors at an image point (Figure 1), we can write the following trigonometric expressions:

$$|p_h|^2 = |p_s|^2 + |p_r|^2 - 2|p_s||p_r|\cos(2\theta), \quad (15)$$
$$|p_m|^2 = |p_s|^2 + |p_r|^2 + 2|p_s||p_r|\cos(2\theta). \quad (16)$$

Equations (15)-(16) relate wavefield quantities, $p_h$ and $p_m$, to a geometric quantity, reflection angle $\theta$. Analysis of these expressions provide sufficient information for complete decompositions of migrated images in components for different reflection angles.
Space-shift imaging condition

Defining $\mathbf{k}_m$ and $\mathbf{k}_h$ as location and offset wavenumber vectors, and assuming $|\mathbf{p}_s| = |\mathbf{p}_r| = s$, where $s$ ($\mathbf{m}$) is the slowness at image locations, we can replace $|\mathbf{p}_m| = |\mathbf{k}_m|/\omega$ and $|\mathbf{p}_h| = |\mathbf{k}_h|/\omega$ in equations (15)-(16):

\[
|\mathbf{k}_h|^2 = 2(\omega s)^2 (1 - \cos 2\theta), \quad (17)
\]
\[
|\mathbf{k}_m|^2 = 2(\omega s)^2 (1 + \cos 2\theta). \quad (18)
\]

Using the trigonometric identity

\[
\cos(2\theta) = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}, \quad (19)
\]

we can eliminate from equations (17)-(18) the dependence on frequency and slowness, and obtain an angle decomposition formulation after imaging by expressing $\tan \theta$ as a function of position and offset wavenumbers ($\mathbf{k}_m, \mathbf{k}_h$):

\[
\tan \theta = \frac{|\mathbf{k}_h|}{|\mathbf{k}_m|}. \quad (20)
\]

We can construct angle-domain common-image gathers by transforming prestack migrated images using equation (20)

\[
R(\mathbf{m}, \mathbf{h}) \rightarrow R(\mathbf{m}, \theta). \quad (21)
\]

In 2D, this transformation is equivalent with a slant-stack on migrated offset gathers. For 3D, this transformation is described in more detail by Fomel (2004) or Sava and Fomel (2005).
Time-shift imaging condition

Using the same definitions as the ones introduced in the preceding subsection, we can re-write equation (18) as

\[ |p_m|^2 = 4s^2 \cos^2 \theta, \quad (22) \]

from which we can derive an expression for angle-transformation after time-shift prestack imaging:

\[ \cos \theta = \frac{|p_m|}{2s}. \quad (23) \]

Relation (23) can be interpreted using ray parameter vectors at image locations (Figure 2). Angle-domain common-image gathers can be obtained by transforming prestack migrated images using equation (23):

\[ R(m, \tau) \Rightarrow R(m, \theta). \quad (24) \]

Equation (23) can be written as

\[ \cos^2 \theta = \frac{|\nabla_m 2\tau|^2}{4s^2(m)} = \frac{\tau_x^2 + \tau_y^2 + \tau_z^2}{s^2(x,y,z)}, \quad (25) \]
where $\tau_x, \tau_y, \tau_z$ are partial derivatives of $\tau$ relative to $x, y, z$. We can rewrite equation (25) as

$$\cos^2 \theta = \frac{\tau_z^2}{s^2(x, y, z)} \left(1 + z_x^2 + z_y^2\right),$$

where $z_x, z_y$ denote partial derivative of coordinate $z$ relative to coordinates $x$ and $y$, respectively. Equation (26) describes an algorithm in two steps for angle-decomposition after time-shift imaging: compute $\cos \theta$ through a slant-stack in $z - \tau$ panels (find a change in $\tau$ with respect to $z$), then apply a correction using the migration slowness $s$ and a function of the structural dips $\sqrt{1 + z_x^2 + z_y^2}$.

**MOVEOUT ANALYSIS**

![Image](image-url)

Figure 3: An image is formed when the Kirchoff stacking curve (dashed line) touches the true reflection response. Left: the case of under-migration; right: over-migration.

We can use the Kirchhoff formulation to analyze the moveout behavior of the time-shift imaging condition in the simplest case of a flat reflector in a constant-velocity medium (Figures 3-5).

The synthetic data are imaged using shot-record wavefield extrapolation migration. Figure 4 shows offset common-image gathers for three different migration slownesses $s$, one of which is equal to the modeling slowness $s_0$. The left column corresponds to the space-shift imaging condition and the right column corresponds to the time-shift imaging condition.

For the space-shift CIGs imaged with correct slowness, left column in Figure 4, the energy is focused at zero offset, but it spreads in a region of offsets when the slowness is wrong. Slant-stacking produces the images in left column of Figure 5.

For the time-shift CIGs imaged with correct slowness, right column in Figure 4, the energy is distributed along a line with a slope equal to the local velocity at
Figure 4: Common-image gathers for space-shift imaging (left column) and time-shift imaging (right column).
Figure 5: Common-image gathers after slant-stack for space-shift imaging (left column) and for time-shift imaging (right column). The vertical line indicates the migration velocity.
the reflector position, but it spreads around this region when the slowness is wrong. Slant-stacking produces the images in the right column of Figure 5.

Let \( s_0 \) and \( z_0 \) represent the true slowness and reflector depth, and \( s \) and \( z \) stand for the corresponding quantities used in migration. An image is formed when the Kirchoff stacking curve \( t(\hat{h}) = 2s\sqrt{z^2 + \hat{h}^2 + 2\tau} \) touches the true reflection response \( t_0(\hat{h}) = 2s_0\sqrt{z_0^2 + \hat{h}^2} \) (Figure 3). Solving for \( \hat{h} \) from the envelope condition \( t'(\hat{h}) = t'_0(\hat{h}) \) yields two solutions:

\[
\hat{h} = 0 \quad (27)
\]

and

\[
\hat{h} = \sqrt{\frac{s_0^2z^2 - s^2z_0^2}{s^2 - s_0^2}} . \quad (28)
\]

Substituting solutions 27 and 28 in the condition \( t(\hat{h}) = t_0(\hat{h}) \) produces two images in the \( \{z, \tau\} \) space. The first image is a straight line

\[
z(\tau) = \frac{z_0 s_0 - \tau}{s} , \quad (29)
\]

and the second image is a segment of the second-order curve

\[
z(\tau) = \sqrt{z_0^2 + \frac{\tau^2}{s^2} - s_0^2} s - \frac{s^2}{s_0^2} . \quad (30)
\]

Applying a slant-stack transformation with \( z = z_1 - \nu \tau \) turns line (29) into a point \( \{z_0 s_0/s, 1/s\} \) in the \( \{z_1, \nu\} \) space, while curve (30) turns into the curve

\[
z_1(\nu) = z_0 \sqrt{1 + \nu^2 \left( s_0^2 - s^2 \right)} . \quad (31)
\]

The curvature of the \( z_1(\nu) \) curve at \( \nu = 0 \) is a clear indicator of the migration velocity errors.

By contrast, the moveout shape \( z(h) \) appearing in wave-equation migration with the lateral-shift imaging condition is \( \text{(Bartana et al., 2005)} \)

\[
z(h) = s_0 \sqrt{\frac{z_0^2}{s^2} + \frac{h^2}{s^2} - s_0^2} . \quad (32)
\]

After the slant transformation \( z = z_1 + h \tan \theta \), the moveout curve (32) turns into the curve

\[
z_1(\theta) = \frac{z_0}{s} \sqrt{s_0^2 + \tan^2 \theta \left( s_0^2 - s^2 \right)} , \quad (33)
\]

which is applicable for velocity analysis. A formal connection between \( \nu \)-parameterization in equation (31) and \( \theta \)-parameterization in equation (33) is given by

\[
\tan^2 \theta = s^2 \nu^2 - 1 , \quad (34)
\]
or

\[ \cos \theta = \frac{1}{\nu s} = \frac{\tau_z}{s}, \quad (35) \]

where \( \tau_z = \frac{\partial \tau}{\partial z} \). Equation (35) is a special case of equation (23) for flat reflectors. Curves of shape (31) and (33) are plotted on top of the experimental moveouts in Figure 5.

**TIME-SHIFT IMAGING IN KIRCHHOFF MIGRATION**

The imaging condition described in the preceding section has an equivalent formulation in Kirchhoff imaging. Traditional construction of common-image gathers using Kirchhoff migration is represented by the expression

\[ R(m, \hat{h}) = \sum_{\hat{m}} \tilde{U} \left[ \hat{m}, \hat{h}, t_s (m, \hat{m} - \hat{h}) + t_r (m, \hat{m} + \hat{h}) \right]. \quad (36) \]

where \( \tilde{U}(\hat{m}, \hat{h}, t) \) is the recorded wavefield at the surface as a function of surface midpoint \( \hat{m} \) and offset \( \hat{h} \) (Figure 6). \( t_s \) and \( t_r \) stand for traveltimes from sources and receivers at coordinates \( \hat{m} - \hat{h} \) and \( \hat{m} + \hat{h} \) to points in the subsurface at coordinates \( m \). For simplicity, the amplitude and phase correction term \( A(m, \hat{m}, \hat{h}) \frac{\partial}{\partial \hat{m}} \) is omitted in equation (36). The time-shift imaging condition can be implemented in Kirchhoff imaging using a modification of equation (36) that is equivalent to equations (6) and (8):

\[ R(m, \tau) = \sum_{m} \sum_{\hat{h}} \tilde{U} \left[ \hat{m}, \hat{h}, t_s (m, \hat{m} - \hat{h}) + t_r (m, \hat{m} + \hat{h}) + 2\tau \right]. \quad (37) \]
Images obtained by Kirchhoff migration as discussed in equation (37) differ from image constructed with equation (36). Relation (37) involves a double summation over surface midpoint \( \hat{m} \) and offset \( \hat{h} \) to produce an image at location \( m \). Therefore, the entire input data contributes potentially to every image location. This is advantageous because migrating using relation (36) different offsets \( \hat{h} \) independently may lead to imaging artifacts as discussed by Stolk and Symes (2004). After Kirchhoff migration using relation (37), images can be converted to the angle domain using equation (23).

**EXAMPLES**

We demonstrate the imaging condition introduced in this paper with the Sigsbee 2A synthetic model (Paffenholz et al., 2002). Figure 7 shows the correct migration velocity and the image created by shot-record migration with wavefield extrapolation using the time-shift imaging condition introduced in this paper. The image in the bottom panel of Figure 7 is extracted at \( \tau = 0 \).

The top row of Figure 8 shows common-image gathers at locations \( x = \{7, 9, 11, 13, 15, 17\} \) km obtained by time-shift imaging condition. As in the preceding synthetic example, we can observe events with linear trends at slopes corresponding to local migration velocity. Since the migration velocity is correct, the strongest events in common-image gathers correspond to \( \tau = 0 \). For comparison, the bottom row of Figure 8 shows common-image gathers at the same locations obtained by space-shift imaging condition. In the later case, the strongest events occur at \( h = 0 \). The zero-offset images \( (\tau = 0 \text{ and } h = 0) \) are identical.

Figure 9 shows the angle-decomposition for the common-image gather at location \( x = 7 \) km. From left to right, the panels depict the migrated image, a common-image gather resulting from migration by wavefield extrapolation with time-shift imaging, the common-image gather after slant-stacking in the \( z - \tau \) plane, and an angle-gather derived from the slant-stacked panel using equation (23).

For comparison, Figure 10 depicts a similar process for a common-image gather at the same location obtained by space-shift imaging. Despite the fact that the offset gathers are completely different, the angle-gathers are comparable showing similar trends of angle-dependent reflectivity.

The top row of Figure 11 shows angle-domain common-image gathers for time-shift imaging at locations \( x = \{7, 9, 11, 13, 15, 17\} \) km. Since the migration velocity is correct, all events are mostly flat indicating correct imaging. For comparison, the bottom row of Figure 11 shows angle-domain common-image gathers for space-shift imaging condition at the same locations in the image.

Finally, we illustrate the behavior of time-shift imaging with incorrect velocity. The top panel in Figure 12 shows an incorrect velocity model used to image the Sigsbee 2A data, and the bottom panel shows the resulting image. The incorrect
Figure 7: Sigsbee 2A model: correct velocity (top) and migrated image obtained by shot-record wavefield extrapolation migration with time-shift imaging condition (bottom).
Figure 8: Imaging gathers at positions $x = \{7, 9, 11, 13, 15, 17\}$ km. Time-shift imaging condition (top row), and space-shift imaging condition (bottom row).
Figure 9: Time-shift imaging condition gather at $x = 7$ km. From left to right, the panels depict the image, the time-shift gather, the slant-stacked time-shift gather and the angle-gather.
Figure 10: Space-shift imaging condition gather at $x = 7$ km: From left to right, the panels depict the image, the space-shift gather, the slant-stacked space-shift gather and the angle-gather.
Figure 11: Angle-gathers at positions $x = \{7, 9, 11, 13, 15, 17\}$ km. Time-shift imaging condition (top row), and space-shift imaging condition (bottom row). Compare with Figure 8.
velocity is a smooth version of the correct interval velocity, scaled by 10% from a depth \( z = 5 \) km downward. The uncollapsed diffractors at depth \( z = 7 \) km clearly indicate velocity inaccuracy.

Figures 13 and 14 show imaging gathers and the derived angle-gathers for time-shift and space-shift imaging at the same location \( x = 7 \) km. Due to incorrect velocity, focusing does not occur at \( \tau = 0 \) or \( h = 0 \) as in the preceding case. Likewise, the reflections in angle-gathers are non-flat, indicating velocity inaccuracies. Compare Figures 9 and 13, and Figures 10 and 14. Those moveouts can be exploited for migration velocity analysis (Biondi and Sava 1999; Sava and Biondi 2004a,b; Clapp et al. 2004).

**DISCUSSION**

As discussed in one of the preceding sections, time-shift gathers consist of linear events with slopes corresponding to the local migration velocity. In contrast, space-shift gathers consist of events focused at \( h = 0 \). Those events can be mapped to the angle-domain using transformations (20) and (23), respectively.

In order to understand the angle-domain mapping, we consider a simple synthetic in which we model common-image gathers corresponding to incidence at a particular angle. The experiment is depicted in Figure 15 for time-shift imaging, and in Figure 16 for space-shift imaging. For this experiment, the sampling parameters are the following: \( \Delta z = 0.01 \) km, \( \Delta h = 0.02 \) km, and \( \Delta \tau = 0.01 \) s.

A reflection event at a single angle of incidence maps in common-image gathers as a line of a given slope. The left panels in Figures 15 and 16 show 3 cases, corresponding to angles of 0°, 20° and 40°. Since we want to analyze how such events map to angle, we subsample each line to 5 selected samples lining-up at the correct slope.

The middle panels in Figures 15 and 16 show the data in the left panels after slant-stacking in \( z - \tau \) or \( z - h \) panels, respectively. Each individual sample from the common-image gathers maps in a line of a different slope intersecting in a point. For example, normal incidence in a time-shift gather maps at the migration velocity \( \nu = 2 \) km/s (Figure 15 top row, middle panel), and normal incidence in a space-shift gather maps at slant-stack parameter \( \tan \theta = 0 \).

The right panels in Figures 15 and 16 show the data from the middle panels after mapping to angle using equations (23) and (20), respectively. All lines from the slant-stack panels map into curves that intersect at the angle of incidence.

We note that all curves for the time-shift angle-gathers have zero curvature at normal incidence. Therefore, the resolution of the time-shift mapping around normal incidence is lower than the corresponding space-shift resolution. However, the storage and computational cost of time-shift imaging is smaller than the cost of equivalent space-shift imaging. The choice of the appropriate imaging condition depends on the imaging objective and on the trade-off between the cost and the desired resolution.
Figure 12: Sigsbee 2A model: incorrect velocity (top) and migrated image obtained by shot-record wavefield extrapolation migration with time-shift imaging condition. Compare with Figure 7.
Figure 13: Time-shift imaging condition gather at $x = 7$ km. From left to right, the panels depict the image, the offset-gather, the slant-stacked gather and the angle-gather. Compare with Figure 9.
Figure 14: Space-shift imaging condition gather at $x = 7$ km. From left to right, the panels depict the image, the offset-gather, the slant-stacked gather and the angle-gather. Compare with Figure 10.
Figure 15: Image-gather formation using time-shift imaging. Each row depicts an event at 0° (top), 20° (middle), and 40° (bottom). Three columns correspond to sub-sampled time-shift gathers (left), slant-stacked gathers (middle), and angle-gathers (right).
Figure 16: Image-gather formation using space-shift imaging. Each row depicts an event at $0^\circ$ (top), $20^\circ$ (middle), and $40^\circ$ (bottom). Three columns correspond to sub-sampled space-shift gathers (left), slant-stacked gathers (middle), and angle-gathers (right).
CONCLUSIONS

We develop a new imaging condition based on time-shifts between source and receiver wavefields. This method is applicable to Kirchhoff, reverse-time and wave-equation migrations and produces common-image gathers indicative of velocity errors. In wave-equation migration, time-shift imaging is more efficient than space-shift imaging, since it only involves a simple phase shift prior to the application of the usual imaging cross-correlation. Disk storage is also reduced, since the output volume depends on only one parameter (time-shift $\tau$) instead of three parameters (space-shift $h$). We show how this imaging condition can be used to construct angle-gathers from time-shift gathers. More research is needed on how to utilize this new imaging condition for velocity and amplitude analysis.

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REFERENCES

Nonhyperbolic moveout

Imaging overturning reflections by Riemannian Wavefield Extrapolation

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ABSTRACT
Correctly propagating waves from overhanging reflectors is crucial for imaging in complex geology. This type of reflections are difficult or impossible to use in imaging using one-way downward continuation, because they violate an intrinsic assumption of this imaging method, i.e. vertical upward propagation of reflection data.

Riemannian wavefield extrapolation is one of the techniques developed to address the limitations of one-way wavefield extrapolation in Cartesian coordinates. This method generalizes one-way wavefield extrapolation to general Riemannian coordinate system. Such coordinate systems can be constructed in different ways, one possibility being construction using ray tracing in a smooth velocity model from a starting plane in the imaged volume. This approach incorporates partially the propagation path into the coordinate system and leaves the balance for the one-way wavefield extrapolation operator. Thus, wavefield extrapolation follows overturning wave paths and extrapolated waves using low-order operators, which makes the extrapolation operation fast and robust.

INTRODUCTION
Imaging of steeply-dipping reflectors, e.g. faults or salt flanks, is a crucial step in seismic imaging of complex geology. In particular, accurate positioning of overhanging salt-flanks influences the quality of migrated images in subsalt regions which are increasingly regarded as the most important targets for seismic exploration.

This challenge for seismic imaging lead to development of many techniques addressing this problem. Among the developed techniques, we can identify:

- **Kirchhoff migration** techniques based on traveltimes computed from overturning rays [Hill et al., 1991, Gray et al., 2001]. Such techniques could be used for imaging of reflections at arbitrary dip angles. However, traveltime computation in complex velocity media requires model approximations, e.g. smoothing of sharp velocity boundaries. Furthermore, Kirchhoff migration using multiple arrivals is possible, but technically challenging.

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• **Reverse-time migration**, based on solutions of the acoustic wave-equation, also has the potential to image reflectors at arbitrary dip angles. Furthermore, such techniques allow for imaging of multiply-reflected waves. However, reverse-time migration is computationally expensive, which limits its usability in practical imaging problems. Nevertheless, despite its large computational cost, reverse-time migration is gaining popularity.

• **Wavefield extrapolation migration** is also employed in imaging steeply-dipping reflectors, despite the intrinsic dip limitation of typical downward continuation operators. However, these techniques have been modified in various ways to allow for imaging of overturning energy. For example, [Hale et al. (1992)] and [Zhang et al. (2006)] employ a succession of downward/upward continuation; [Rietveld and Berkhout (1994)] and [Shan and Biondi (2004)] use tilted coordinates to bring the direction of extrapolation closer to the direction of wave propagation; [Brandsberg-Dahl and Etgen (2003)] use extrapolation along beams to achieve even tighter proximity of the directions of extrapolation and wave propagation; [Sava and Fomel (2005)] use one-way extrapolation in general (Riemannian) coordinate systems.

This paper concentrates on using Riemannian wavefield extrapolation (RWE) for imaging reflectors with high dip angles. The basic characteristics of RWE recommend it as a good candidate for imaging of steeply-dipping reflectors: like a Kirchhoff technique, the (overturning) Riemannian coordinate system allows extrapolation of waves along their natural direction of propagation; like a wavefield extrapolation technique, RWE allows for extrapolation of all branches of the wavefield, thus making used of all multiple-paths of extrapolated wavefields. Extrapolation in Cartesian coordinates, including tilted coordinates, and extrapolation along beams, represent special cases of RWE for particular choices of the coordinate system. Coordinate systems for RWE can be constructed by ray tracing or by other approaches based on alternative criteria, e.g. conformal maps with topography [Shragge and Sava (2005)].

This paper demonstrates the applicability of RWE to the problem of imaging steeply-dipping reflectors, in particular (overhanging) salt flanks. In addition to accurate implementation of extrapolation, a challenge for RWE is represented by the construction of the coordinate system that is appropriate for imaging of particular reflectors. Thus, a large fraction of this paper is dedicated to coordinate-system construction methods.

This paper is organized as follows: we begin with a brief review of Riemannian wavefield extrapolation, then describe alternatives for the construction of coordinate systems supporting RWE, and demonstrate the technique with applications to imaging of overhanging salt flank for synthetic salt modeled data.
RIEMANNIAN WAVEFIELD EXTRAPOLATION

Riemannian wavefield extrapolation (Sava and Fomel, 2005) generalizes solutions to the Helmholtz equation of the acoustic wave-equation

$$\Delta U = -\omega^2 s^2 U ,$$

(1)

to non-Cartesian coordinate systems, such that extrapolation is not performed strictly in the downward direction. In equation 4, $s$ is slowness, $\omega$ is temporal frequency, and $U$ is a monochromatic acoustic wave.

Assume that we describe the physical space in Cartesian coordinates $x$, $y$ and $z$, and that we describe a Riemannian coordinate system using coordinates $\xi$, $\eta$ and $\zeta$ related through a generic mapping

$$x = x(\xi, \eta, \zeta) ,$$

(2)

$$y = y(\xi, \eta, \zeta) ,$$

(3)

$$z = z(\xi, \eta, \zeta) ,$$

(4)

which allows us to compute derivatives of the Cartesian coordinates relative to the Riemannian coordinates.

Following the derivation of Sava and Fomel (2005), the acoustic wave-equation in Riemannian coordinates can be written as:

$$c_{\zeta \zeta} \frac{\partial^2 U}{\partial \zeta^2} + c_{\xi \xi} \frac{\partial^2 U}{\partial \xi^2} + c_{\eta \eta} \frac{\partial^2 U}{\partial \eta^2} + c_{\xi \eta} \frac{\partial^2 U}{\partial \xi \partial \eta} = - (\omega s)^2 U .$$

(5)

where coefficients $c_{ij}$ are spatially-variable functions of the coordinate system and can be computed numerically for any given coordinate system using the mappings 2-4.

The acoustic wave-equation in Riemannian coordinates 5 ignores the influence of first order terms present in a more general acoustic wave-equation in Riemannian coordinates. This approximation is justified by the fact that, according to the theory of characteristics for second-order hyperbolic equations (Courant and Hilbert, 1989), the first-order terms affect only the amplitude of the propagating waves.

From equation 5 we can derive a dispersion relation of the acoustic wave-equation in Riemannian coordinates

$$- c_{\zeta \zeta} k_{\zeta}^2 - c_{\xi \xi} k_{\xi}^2 - c_{\eta \eta} k_{\eta}^2 - c_{\xi \eta} k_{\xi} k_{\eta} = - (\omega s)^2 ,$$

(6)

where $k_{\zeta}$, $k_{\xi}$ and $k_{\eta}$ are wavenumbers associated with the Riemannian coordinates $\zeta$, $\xi$ and $\eta$. For one-way wavefield extrapolation, we need to solve the quadratic equation 6 for the wavenumber of the extrapolation direction $k_{\zeta}$, and select the solution with the appropriate sign for the desired extrapolation direction:

$$k_{\zeta} = \frac{\sqrt{(\omega s)^2 - c_{\xi \xi} k_{\xi}^2 - c_{\eta \eta} k_{\eta}^2 - c_{\xi \eta} k_{\xi} k_{\eta}}}{c_{\zeta \zeta}}.$$

(7)
The coordinate system coefficients $c_{ij}$ and the extrapolation slowness $s$ can be combined to form a reduced set of parameters. In 2D, for example, all coordinate-system coefficients can be represented by 2 parameters, $a$ and $b$. Further extensions and implementation details of equation 7 are described by Sava and Fomel (2007).

Extrapolation using equation 7 implies that the coefficients defining the medium and coordinate system are not changing spatially. In this case, we can perform extrapolation using a simple phase-shift operation

$$U_{r+\Delta r} = U_r e^{ik_r \Delta r},$$

where $U_{r+\Delta r}$ and $U_r$ represent the acoustic wavefield at two successive extrapolation steps, and $k_r$ is the extrapolation wavenumber defined by equation 7.

For media with variability of the coefficients $c_{ij}$ due to either velocity variation or focusing/defocusing of the coordinate system, we cannot use in extrapolation the wavenumber computed directly using equation 7. Like for the case of extrapolation in Cartesian coordinates, we can approximate the wavenumber $k_r$ using series expansions relative to coefficients $c_{ij}$ present in the dispersion relation 7. Such approximations can be implemented in the space-domain, in the Fourier domain or in mixed space-Fourier domains (Sava and Fomel 2007).

**COORDINATE SYSTEMS**

Riemannian wavefield extrapolation operates in coordinate systems that may or may not be defined according to the model used for imaging. As indicated earlier, there are several options for constructing such coordinate systems, but the solution selected for the case described in this paper uses ray tracing. In this case, the coordinate system is semi-orthogonal, i.e. the extrapolation direction is orthogonal to the other two directions defining a 3D coordinate system. Cartesian coordinates are special cases of Riemannian coordinates constructed by tracing rays orthogonal to a flat surface in constant velocity.

The accuracy of one-way wavefield extrapolation operators maximizes in the direction of extrapolation (vertical for downward continuation; along rays for Riemannian coordinate systems constructed by ray tracing). In addition, for steeply dipping reflectors, the angle of reflection is likely to be relatively small since this reflector is illuminated from a large distance using a small limited acquisition aperture.

It is thus desirable to construct a coordinate system that minimizes the angles between the extrapolation direction, and the directions of wave propagation and normal to the imaged reflectors. One way to achieve this goal is to construct the coordinate system by tracing rays orthogonal to an imaginary line (plane in 3D) located behind the overhanging reflector.

These ideas are illustrated with a model based on the Sigsbee 2A synthetic (Paffenholz et al. 2002) extended vertically and horizontally to allow diving waves from...
the overhanging salt flank to arrive at the surface. Figure 1(a) shows an example of Riemannian coordinates constructed from ray tracing from behind the salt flank reflector. For comparison, Figure 1(b) shows a Cartesian coordinate system tilted relative to the vertical direction to minimize the angle between the extrapolation direction and the normal to the reflector.

![Figure 1: Riemannian coordinate system (a) and tilted Cartesian coordinate system (b).](jse2006RWEImagingOverturningReflections/sigsbee cos1,cos2)

As indicated in the preceding section, we can describe Riemannian coordinate systems with several coefficients incorporating all the information about the coordinate system shape and the extrapolation slowness. For this 2D example, there are
two coefficients, $a$ and $b$ depicted in Figures 2(a) and 2(b) for the Riemannian coordinate system and in Figures 3(a) and 3(b) for the tilted Cartesian coordinate system. The plots depict $a$ and $b$ function of the Riemannian coordinates, $\tau$ and $\gamma$. $\tau$ has time units and it represents the extrapolation direction, and $\gamma$ is non-dimensional and represents an index of the rays shot from the linear origin behind the imaged reflector.

Coefficient $a$ describes the ratio of the extrapolation velocity to the velocity used for ray tracing, and coefficient $b$ describes the focusing/defocusing of the coordinate system. In both cases, coefficient $a \neq 1$ since the velocity used for extrapolation is different from the velocity used for the coordinate system. For the Cartesian coordinate system, coefficient $b$ is a constant, as depicted in Figure 3(b).

Figures 2(c) and 3(c) depict the acquisition surface and the salt body outline mapped in Riemannian and tilted Cartesian coordinates, respectively. We can observe that in Riemannian coordinates, the overhanging salt flank is nearly orthogonal to the extrapolation direction, unlike its layout in tilted Cartesian coordinates. Therefore, imaging accuracy for such reflectors can be achieved in Riemannian coordinates with lower order extrapolation kernels than in the case of tilted Cartesian coordinates, as demonstrated in the following section.

Figure 2: Riemannian coordinate system coefficients, (a) and (b), and outline of acquisition geometry and salt body (c).
Figure 3: Tilted Cartesian coordinate system coefficients, (a) and (b), and outline of acquisition geometry and salt body (c).
EXAMPLE

The Riemannian wavefield extrapolation imaging procedure is illustrated with the synthetic example introduced in the preceding section. Exploding reflector data are modeled from all edges of the salt body and recorded at all locations along the surface. The data are modeled using time-domain finite-differences. No attempt is made to suppress the multiples, thus some of them are present in the migrated images.

Figures 4(a)-4(b) depict wavefields at 5 and 10 seconds from the exploding reflector moment, respectively. As expected, the wavefield originating on the overhanging salt flank dives and then returns to the surface (Figure 5). Other parts of the wavefield either propagate straight up to the surface, or propagate away from it and are not recorded. The imaged components of the recorded wavefield are those overturning to the surface at locations intersected by the coordinate system.

The overturned data collected at the surface are imaged in Riemannian and tilted Cartesian coordinates using the systems depicted in Figures 1(a)-1(b), characterized by coefficients $a$ and $b$ depicted in Figures 2(a)-2(b) and Figures 3(a)-3(b), respectively.

Figures 6(a)-6(b) show the migrated image obtained by wavefield extrapolation in Riemannian coordinates. Panel 6(a) depicts the image in the Riemannian space, and panel 6(b) depicts the same image after mapping to vertical Cartesian coordinates. Both overhanging reflectors, on opposite sides of the salt body, are imaged correctly demonstrating successful imaging with overturning waves in Riemannian coordinates.

Figures 7(a)-7(b) show analogous images obtained by extrapolation in tilted Cartesian coordinates. Imaging is performed using the same extrapolation as the one used for imaging in Riemannian coordinates. The only difference is the coordinate system, therefore all image differences are caused strictly by the coordinate system and not by the extrapolation or imaging operators.

Unlike for the preceding example, the overhanging salt flanks are not positioned correctly, as seen by comparing Figures 6(a) and 7(a). The reason for the inaccurate reflector positioning is the limited accuracy of the extrapolation operator at high angles relative to the extrapolation direction. While the Riemannian coordinate system has the flexibility to minimize the angle between the extrapolation direction and the direction of wave propagation, the tilted Cartesian coordinate system cannot do that, thus requiring wavefield extrapolation at high angles. In this example, modifying the tilt angle of the Cartesian coordinate system does not help since propagation at high angles occurs both in the vicinity of the reflector and close to the acquisition surface. Furthermore, the Riemannian coordinate system represents the natural direction of wave propagation, unlike the tilted Cartesian coordinate system which is artificial and has no physical relation to the propagating waves.
Figure 4: Wavefield from the overhanging salt flank at $t = 5,10$ s from the moment of exploding reflectors on the salt body.
Figure 5: Recorded data at the surface. Overturning energy is recorded from $x = -20$ km to $x = 0$ km at $t = 15 - 25$ s.
Figure 6: Migrated images using wavefield extrapolation in Riemannian coordinates with a Fourier finite-differences (F15) kernel. Panel (a) depicts the image in Riemannian coordinates, and panel (b) depicts the same image mapped to vertical Cartesian coordinates.
Figure 7: Migrated images using wavefield extrapolation in tilted Cartesian coordinates with a Fourier finite-differences (F15) kernel. Panel (a) depicts the image in tilted Cartesian coordinates, and panel (b) depicts the same image mapped to vertical Cartesian coordinates.
CONCLUSIONS

This paper demonstrates the applicability of Riemannian wavefield extrapolation to the problem of imaging overhanging salt flanks. Imaging such reflectors using one-way wavefield extrapolation in Cartesian coordinates is impractical since waves propagate partially down, partially up. A possible solution to this problem consists of using tilted Cartesian coordinate systems. This procedure partially reduces the angle between the direction of wave propagation and the direction of extrapolation. However, even in this coordinate framework, waves need to be extrapolated at high angles up to 90° which degrades the imaging accuracy.

In contrast, wavefield extrapolation in Riemannian coordinates has the flexibility to follow closely the paths of wave propagation. Therefore, the relative angle between the direction of extrapolation and the direction of wave propagation is much smaller than in the case of extrapolation in tilted Cartesian coordinates, thus improving imaging accuracy.

Overturning reflections can, in principle, be imaged using Kirchhoff migration. However, this imaging procedure has difficulty producing accurate images in complex geology characterized by wave multipathing and sharp velocity variation. In contrast, imaging overturning reflections using Riemannian wavefield extrapolation benefits from all characteristics of one-way wavefield extrapolation, i.e. stability across boundaries between media with large velocity variation, multipathing, etc.

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Stereographic imaging condition for wave-equation migration

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ABSTRACT
Imaging under the single-scattering approximation consists of two steps: wavefield reconstruction of source and receiver wavefields from simulated and recorded data, respectively, and imaging from the extrapolated wavefields of the locations where reflectors occur. Conventionally, the imaging condition indicates the presence of reflectors when propagation times of reflections in the source and receiver wavefields match. The main drawback of conventional cross-correlation imaging condition is that it ignores the local spatial coherence of reflection events and relies only on their propagation time. This leads to interference between unrelated events that occur at the same time. Sources of cross-talk include seismic events corresponding to different seismic experiments, or different propagation paths, or different types of reflections (primary or multiple) or different wave modes (P or S). An alternative imaging condition operates on the same extrapolated wavefields, but cross-correlation takes place in a higher-dimensional domain where seismic events are separated based on their local space-time slope. Events are matched based on two parameters (time and local slope), thus justifying the name “stereographic” for this imaging condition. Stereographic imaging attenuates wavefield cross-talk and reduces imaging artifacts compared with conventional imaging. Applications of the stereographic imaging condition include simultaneous imaging of multiple seismic experiments, multiple attenuation in the imaging condition, and attenuation of cross-talk between multiple wavefield branches or between multiple wave modes.

INTRODUCTION
Conventional depth migration consists of two steps: wavefield reconstruction of seismic wavefields at all locations in the imaging volume from data recorded on the acquisition surface, and imaging used to extract reflectivity information from wavefields reconstructed from the sources and receivers. Accurate imaging requires accurate implementation of both steps. Recent seismic imaging research places larger emphasis on wavefield extrapolation than on imaging, partly due to the larger computational cost of extrapolation relative to imaging.

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This paper concentrates on the imaging condition assuming that wavefield extrapolation is performed in a sufficiently accurate velocity model. The imaging condition is often implemented as a cross-correlation of source and receiver wavefields extrapolated from the acquisition surface \cite{Claerbout1985}. The reason for this choice is that conventional cross-correlation imaging is fast and robust, producing good images in complex environments. The alternative deconvolution imaging condition is not discussed in this paper.

Conventional imaging condition operates in a simple way: source and receiver wavefields are probed to determine the locations where they match, i.e. where the traveltime of events forward-propagated from the source and backward-propagated from the receivers are equal. This is usually achieved by extracting the zero-lag of the temporal cross-correlation between the two wavefields computed at every location in the image. However, this imaging condition ignores the structure of the analyzed seismic wavefields, i.e. the imaging condition does not use the local space-time coherence of the reflected wavefields. This characteristic is contrary to conventional analysis of space-time kinematic coherence of seismic data, which is one of the most important attributes employed in their analysis.

The consequence of this deficiency is that different seismic events present in the extrapolated wavefields interfere with one-another leading to artifacts in seismic images. This interference, also known as cross-talk, occurs between unrelated events which should not contribute to the formed image. It is often possible to identify events that occur at the same time, although they describe different propagation paths in the subsurface. As a consequence, such unrelated events appear as real reflections due to the imaging condition and not due to a geological cause.

This paper presents an extension of the conventional imaging condition. This extension is designed to exploit the local space-time coherence of extrapolated wavefields. Different seismic events are matched both function of propagation time and a local coherence attributes, e.g. local slope measured function of position and time. Therefore, events with different propagation paths are differentiated from one-another, although their propagating time to a given point in the subsurface may be identical \cite{Stolk2004}. This property can be used to suppress artifacts due to cross-talk and generate cleaner seismic images.

**CONVENTIONAL IMAGING CONDITION**

Under the single scattering (Born) approximation, seismic migration consists of two components: wavefield reconstruction and imaging.

Wavefield reconstruction forms solutions to the considered (acoustic) wave-equation with recorded data as boundary condition. We can consider many different numeric solutions to the acoustic wave-equation, which are distinguished, for example, by implementation domain (space-time, frequency-wavenumber, etc.) or type of numeric solution (differential, integral, etc.). Irrespective of numeric implementation, we re-
construct two wavefields, one forward-propagated from the source and one backward-propagated from the receiver locations. Those wavefields can be represented as four-dimensional objects function of position in space $\mathbf{x} = (x, y, z)$ and time $t$

\[
u_s = \nu_s(\mathbf{x}, t) \tag{1}
\]

\[
u_r = \nu_r(\mathbf{x}, t) \tag{2}
\]

where $\nu_s$ and $\nu_r$ denote source and receiver wavefields. For the remainder of this paper, we can assume that the two wavefields have been reconstructed with one of the numerical methods mentioned earlier.

The second migration component is the imaging condition which is designed to extract from the extrapolated wavefields ($\nu_s$ and $\nu_r$) the locations where reflectors occur in the subsurface. The image $r$ can be extracted from the extrapolated wavefields by evaluating the match between the source and receiver wavefields at every location in the subsurface. The wavefield match can be evaluated using an extended imaging condition (Sava and Fomel, 2005, 2006), where image $r$ represents an estimate of the similarity between the source and receiver wavefields in all 4 dimensions, space ($\mathbf{x}$) and time ($t$):

\[
r(\mathbf{x}, \lambda, \tau) = \int \nu_s(\mathbf{x} - \lambda, t - \tau) \nu_r(\mathbf{x} + \lambda, t + \tau) \, dt. \tag{3}
\]

The quantities $\lambda$ and $\tau$ represent the spatial and temporal cross-correlation lags between the source and receiver wavefields. The source and receiver wavefields are coincident (i.e. form an image) if the local cross-correlation between the source and receiver wavefields maximizes at zero-lag on all four dimensions. Other extended imaging conditions (Rickett and Sava, 2002; Biondi and Symes, 2004) represent special cases of the extended imaging condition corresponding to horizontal $\lambda = (\lambda_x, \lambda_y, 0)$, or vertical $\lambda = (0, 0, \lambda_z)$ space lags, respectively. The conventional imaging condition (Claerbout, 1985) is also a special case of the extended imaging condition corresponding to zero cross-correlation lag in space ($\lambda = 0$) and time ($\tau = 0$):

\[
r(\mathbf{x}) = \int \nu_s(\mathbf{x}, t) \nu_r(\mathbf{x}, t) \, dt. \tag{4}
\]

The four-dimensional cross-correlation maximizes at zero lag if the source and receiver wavefields are correctly reconstructed. If this is not true, either because we are using an approximate extrapolation operator (e.g. one-way extrapolator with limited angular accuracy), or because the velocity used for extrapolation is inaccurate, the four-dimensional cross-correlation does not maximize at zero lag and part of the cross-correlation energy is smeared over space and time lags ($\lambda$ and $\tau$). Therefore, extended imaging conditions can be used to evaluate imaging accuracy, for example by decomposition of reflectivity function of scattering angle at every image location (Sava and Fomel, 2003; Biondi and Symes, 2004; Sava and Fomel, 2006). Angle-domain images carry information useful for migration velocity analysis (Biondi and Sava, 1999; Sava and Biondi, 2004a,b; Shen et al., 2005), or for amplitude analysis.
The conventional imaging condition \(2\) is the focus of this paper. As discussed above, assuming accurate extrapolation, this imaging condition should produce accurate images at zero cross-correlation lags. However, this conclusion does not always hold true, as illustrated next.

Figures 1(a) and 1(b) represent a simple model of constant velocity with a horizontal reflector. Data in this model are simulated from 3 sources triggered simultaneously at coordinates \(x = 600, 1000, 1200\) m. Using the standard imaging procedure outlined in the preceding paragraphs, we can reconstruct the source and receiver wavefields, \(u_s\) and \(u_r\), and apply the conventional imaging condition equation 2 to obtain the image in Figure 3(a). The image shows the horizontal reflector superposed with linear artifacts of comparable strength.

Figures 2(a) and 2(b) represent another simple model of spatially variable velocity with a horizontal reflector. Data in this model are simulated from a source located at coordinate \(x = 1000\) m. The negative Gaussian velocity anomaly present in the velocity model creates triplications of the source and receiver wavefields. Using the same standard imaging procedure outlined in the preceding paragraphs, we obtain the image in Figure 5(a). The image also shows the horizontal reflector superposed with complex artifacts of comparable strength.

In both cases discussed above, the velocity model is perfectly known and the acoustic wave equation is solved with the same finite-difference operator implemented in the space-time domain. Therefore, the artifacts are caused only by properties of the conventional imaging condition used to produce the migrated image and not by inaccuracies of wavefield extrapolation or of the velocity model.

The cause of artifacts is cross-talk between events present in the source and receiver wavefields, which are not supposed to match. For example, cross-talk can occur between wavefields corresponding to multiple sources, as illustrated in the example shown in Figures 1(a)-1(b) multiple branches of a wavefield corresponding to one source, as illustrate in the example shown in Figures 2(a)-2(b) events that correspond to multiple reflections in the subsurface, or multiple wave modes of an elastic wavefield, for example between PP and PS reflections, etc.

**STEREOGRAPHIC IMAGING CONDITION**

One possibility to remove the artifacts caused by the cross-talk between inconsistent reflection events is to modify the imaging condition to use more than one attribute for matching the source and receiver wavefields. For example, we could use the time and slope to match events in the wavefield, thus distinguishing between unrelated events that occur at the same time (Figure 4).

A simple way of decomposing the source and receiver wavefields function of local
slope at every position and time is by local slant-stacks at coordinates $x$ and $t$ in the four-dimensional source and receiver wavefields. Thus, we can write the total source and receiver wavefields ($u_s$ and $u_r$) as a sum of decomposed wavefields ($w_S$ and $w_R$):

$$u_s(x, t) = \int w_S(x, p, t) \, dp$$

$$u_r(x, t) = \int w_R(x, p, t) \, dp .$$

Here, the three-dimensional vector $p$ represents the local slope function of position and time. Using the wavefields decomposed function of local slope, $w_S$ and $w_R$, we can design a stereographic imaging condition which cross-correlates the wavefields in the decomposed domain, followed by summation over the decomposition variable:

$$r(x) = \int \int w_S(x, p, t) \, w_R(x, p, t) \, dp \, dt .$$

Correspondence between the slopes $p$ of the decomposed source and receiver wavefields occurs only in planes dipping with the slope of the imaged reflector at every location in space. Therefore, an approximate measure of the expected reflector slope is required for correct comparison of corresponding reflection data in the decomposed wavefields. The choice of the word “stereographic” for this imaging condition is analogous to that made for the velocity estimation method called stereotomography (Billette and Lambare, 1997; Billette et al., 2003) which employs two parameters (time and slope) to constrain traveltime seismic tomography.

For comparison with the stereographic imaging condition, the conventional imaging condition can be reformulated using the wavefield notation as follows:

$$r(x) = \int \left[ \int w_S(x, p, t) \, dp \right] \left[ \int w_R(x, p, t) \, dp \right] \, dt .$$

The main difference between imaging conditions and is that in one case we are comparing independent slope components of the wavefields separated from one-another, while in the other case we are comparing a superposition of them, thus not distinguishing between waves propagating in different directions. This situation is analogous to that of reflectivity analysis function of scattering angle at image locations, in contrast with reflectivity analysis function of acquisition offset at the surface. In the first case, waves propagating in different directions are separated from one-another, while in the second case all waves are superposed in the data, thus leading to imaging artifacts (Stolk and Symes, 2004).

Figure shows the image produced by stereographic imaging of the data generated for the model depicted in Figures 1(a)-1(b), and Figure 5(b) shows the similar image for the model depicted in Figures 2(a)-2(b). Images 3(b) and 5(b) use the same source receiver wavefields as images 3(a) and 5(a) respectively. In both cases, the cross-talk artifacts have been eliminated by the stereographic imaging condition.
EXAMPLE

The stereographic imaging condition is illustrated with an example derived from the Sigsbee 2A dataset (Paffenholz et al., 2002). Using the model in Figure 6(g), two shots are simulated by wavefield extrapolation modeling, Figures 6(a)-6(c), and a third shot is synthesized by summing the two shots together, Figure 6(e). Migration with conventional imaging condition of the three shots produces the images in Figures 6(b)-6(f). The two shots independently illuminate different parts of the model, Figures 6(b)-6(d), while the third composite shot illuminates both sides of the image, Figure 6(f). The image produced by the composite shot is populated with artifacts due to the cross-talk between the wavefields originating at the two shot locations.

Figure 6(h) shows the image obtained by imaging the composite shot, Figure 6(e), using the stereographic imaging condition. The image is free of artifacts and shows reflectors extending over the entire image, as would be expected for illumination from two shots at different locations. In this case, the stereographic imaging condition needs to take into account the local dip of the image. Since we cannot know the reflector dip prior to the application of the imaging condition, we need to loop over a range of possible dip angles and decompose the wavefields locally for all possible slope combinations. Thus, the stereographic imaging procedure matches the dip of wavefield components in local windows around every image point. Assuming that the local geologic dip is known, at least approximately, we could consider looping over a small range of local dips, thus decreasing the cost of the imaging condition. This approach was not used for the examples shown in this paper and remains to be investigated by future research.

DISCUSSION

The imaging procedure described in this paper requires additional steps that add to the computational cost of imaging. Furthermore, there are more parameters that need to be chosen. For example, if we use local slant-stacks for local decomposition, we need to decide how many local slopes we should use, how finely we need to sample the slope parameters, how finely in space should we apply slant-stacking of the source/receiver wavefields, etc. The number of local slopes used for the imaging condition depends on wavefield sampling in space and time in order to avoid aliasing. Those challenges remain to be addressed by future research.

In all examples described in this paper, the local windows have simple rectangular shape. However, more sophisticated window types (e.g. Gaussian) are possible alternatives and might improve the quality and efficiency of the method.

We can consider tuning the stereographic imaging condition for specific applications. In current implementation, only image components with spatial coherence (e.g. reflectors) generate wavefields with spatial coherence. Diffractions, for example, do not fit this description and, thus, are removed from the image by the imaging condi-
tion. This can be seen both as a feature or as a drawback depending on the type of imaging target.

CONCLUSIONS

Conventional imaging conditions based on cross-correlation of extrapolated wavefields do not take into account the local spatial coherence of reflection events. Events are matched based on their propagation times, which leads to cross-talk between unrelated events. The stereographic imaging condition introduced in this paper operates on seismic wavefields that are first decomposed function of their local slope in space and time. Events are matched based on two parameters (time and local slope), which separates unrelated events and eliminates cross-talk. Higher imaging accuracy is achieved at the expense of larger computational cost. Applications include simultaneous imaging of different seismic experiments (shots), multiple attenuation in the imaging condition, etc.

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Figure 1: Constant velocity model (a), reflectivity model (b), data (c) and shot locations at $x = 600, 1000, 1200$ m.)
Figure 2: Velocity model with a negative Gaussian anomaly (a), reflectivity model (b), data (c) and shot location at $x = 1000$ m.)
Figure 3: Images obtained for the model in Figures 1(a)-1(c) using the conventional imaging condition (a) and the stereographic imaging condition (b).

Figure 4: Comparison of conventional imaging (a) and stereographic imaging (b).
Figure 5: Images obtained for the model in Figures 2(a)-2(c) using the conventional imaging condition (a) and the stereographic imaging condition (b).
Figure 6: Data corresponding to shots located at coordinates $x = 16$ kft (a), $x = 24$ kft (c), and the sum of data corresponding to both shot locations (e). Image obtained by conventional imaging condition for the shots located at coordinates $x = 16$ kft (b), $x = 24$ kft (d) and the sum of data for both shots (f). Velocity model extracted from the Sigsbee 2A model (g) and image from the sum of the shots located at $x = 16$ kft and $x = 24$ kft obtained using the stereographic imaging condition (h).
Numeric implementation of
wave-equation migration velocity analysis operators

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ABSTRACT
Wave-equation migration velocity analysis (MVA) is a technique similar to wave-equation tomography because it is designed to update velocity models using information derived from full seismic wavefields. On the other hand, wave-equation MVA is similar to conventional, traveltime-based MVA because it derives the information used for model updates from properties of migrated images, e.g. focusing and moveout. The main motivation for using wave-equation MVA is derived from its consistency with the corresponding wave-equation migration, which makes this technique robust and capable of handling multipathing characterizing media with large and sharp velocity contrasts. The wave-equation MVA operators are constructed using linearizations of conventional wavefield extrapolation operators, assuming small perturbations relative to the background velocity model. Similarly to typical wavefield extrapolation operators, the wave-equation MVA operators can be implemented in the mixed space-wavenumber domain using approximations of different orders of accuracy.

As for wave-equation migration, wave-equation MVA can be formulated in different imaging frameworks, depending on the type of data used and image optimization criteria. Examples of imaging frameworks correspond to zero-offset migration (designed for imaging based on focusing properties of the image), survey-sinking migration (designed for imaging based on moveout analysis using narrow-azimuth data) and shot-record migration (also designed for imaging based on moveout analysis, but using wide-azimuth data).

The wave-equation MVA operators formulated for the various imaging frameworks are similar because they share common elements derived from linearizations of the single square-root equation. Such operators represent the core of iterative velocity estimation based on diffraction focusing or semblance analysis, and their applicability in practice requires efficient and accurate implementation. This tutorial concentrates strictly on the numeric implementation of those operators and not on their use for iterative migration velocity analysis.

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INTRODUCTION

Accurate wave-equation depth imaging requires accurate knowledge of a velocity model. The velocity model is used in the process of wavefield reconstruction, irrespective of the method used to solve the acoustic wave-equation, e.g. by integral methods (Kirchhoff migration), or by differential/spectral methods (migration by wavefield extrapolation and reverse-time migration).

Generally-speaking, there are two possible strategies for velocity estimation from surface seismic data in the context of wavefield depth migration. The two strategies differ by the domain in which the information used to update the velocity model is collected. The first strategy is formulated in the data space, prior to migration, and it involves matching of recorded and simulated data using an approximate (background) velocity model. Techniques in this category are known by the name of tomography (or inversion). The second strategy is formulated in the image space, after migration, and it involves measuring and correcting image features that indicate model inaccuracies. Techniques in this category are known as migration velocity analysis (MVA), since they involve migrated images and not the recorded data directly.

Tomography and migration velocity analysis can be implemented in various ways that can be classified based on the carrier of information from the data or image to the velocity model. Thus, we can distinguish between ray-based methods and wave-based methods. This terminology is applicable to both tomography and migration velocity analysis. For ray-based methods, the carrier of information are wide-band rays traced using a background velocity model from picked events in the data (or image). Methods in this category are known as traveltime tomography (Bishop et al., 1985) and traveltime MVA, sometimes described as image-space traveltime tomography (Stork, 1992; Al-Yahya, 1987; Fowler, 1988; Etgen, 1990; Chavent and Jacewitz, 1995; Clement et al., 2001; Chauris et al., 2002a,b; Billette et al., 2003; Lambare et al., 2004; Clapp et al., 2004). For wave-based methods, the carrier of information are band-limited wavefields constructed using a background velocity model. Methods in this category are known as wave-equation tomography (Gauthier et al., 1986; Tarantola, 1987; Mora, 1989; Woodward, 1992; Pratt, 1999; 2004), and wave-equation MVA (Biondi and Sava, 1999; Sava and Biondi, 2004a,b; Shen et al., 2003; Albertin et al., 2006b; Maharramov and Albertin, 2007). This paper concentrates on methods from the latter category.

The volume of information used for model updates using wave-based methods is at least one order of magnitude larger than the volume of information used for ray-based methods. Thus, a fundamental question we should ask is what is gained by using wave-based methods over ray-based methods. First, modern imaging applications using wave-based methods (downward continuation or reverse-time extrapolation) require consistent velocity estimation methods which interact with model in the same frequency band as the migration methods. Second, wave-based methods are robust (i.e. stable) for models with large and sharp velocity variations (e.g. salt). Third, wave-based methods describe naturally all propagation paths, thus they can easily
handle multi-pathing characterizing wave propagation in media with large velocity variations.

Wave-equation tomography and wave-equation MVA have both similarities and differences. Wave-equation tomography uses the advantage that the residual used for velocity updating is obtained by a direct comparison between recorded and measured data. In contrast, wave-equation MVA uses the property that the residual used for velocity updating is obtained by a comparison between a reference image and an improved version of it. On the other hand, wave-equation tomography has the disadvantage that the kinematics of events in the data domain are more complex than in the image domain. In addition, the dimensionality of the space in which to evaluate the misfit between recorded and simulated data is higher too, potentially making a comparison more complex. Also, wave-equation MVA optimizes directly the desired end product, i.e. the migrated image, which potentially makes this technique more “interpretive” and less of a computational “black-box”.

One important component of MVA methods is what type of measurement on migrated images is used to evaluate its deficiencies. Although strictly related to one-another, we can describe two kinds of information available to quantify image quality. First is focusing analysis, which evaluates whether point-like events, e.g. fault truncations, are focused in migrated images at their correct position. Image enhancement for wave-equation MVA can be formulated purely based on this type of information, which makes the techniques fast since it only operates with zero-offset data (Sava et al., 2005). Second is moveout analysis, which is the case for all conventional MVA techniques, whether using rays or waves. In this case, we can formulate wave-equation MVA based on analysis of moveout of common-image gathers using velocity scans (Biondi and Sava, 1999; Sava and Biondi, 2004a, b; Soubaras and Gratacos, 2007), or based on analysis of differential semblance of nearby traces in similar common-image gathers (Shen et al., 2005).

Moveout analysis requires construction of common-image gathers (CIGs) characterizing the dependence of reflectivity function of various parameters used to parametrize multiple experiments used for imaging. There are two main alternatives for common-image gather construction. First, we can construct offset-domain CIGs (Yilmaz and Chambers, 1984), when reflectivity depends on source-receiver offset on the acquisition surface, which is a data parameter. Second, we can construct angle-domain CIGs (de Bruin et al., 1990; Prucha et al., 1999; Mosher and Foster, 2000; Rickett and Sava, 2002; Xie and Wu, 2002; Sava and Fomel, 2003; Soubaras, 2003; Fomel, 2004; Biondi and Symes, 2004), when reflectivity depends on the angles of incidence at the reflection point, which is an image parameter. For wave-equation migration, offset-domain CIGs are not a practical solution, since the information from multiple offsets (or multiple seismic experiments) mixes in the process of migration. Furthermore, angle-domain CIGs suffer from fewer artifacts than offset-domain CIGs, due to the fact that reflectivity parametrization for angle-gathers occurs after wavefield reconstruction in the subsurface, as oppose to offset-gathers when reflectivity parametrization is related to data parameters (Stolk and Symes, 2004).
Both wave-equation tomography and wave-equation MVA methods are based on a fundamental “small perturbation” assumption, which requires a reasonably-good starting model. This requirement represents a drawback which is responsible for the main difficulty of methods in both categories. For wave-equation tomography or inversion, we can update models based on differences between recorded and simulated data. If the starting background model is not accurate enough, we run the risk of subtracting wavefields corresponding to different events. Likewise, for wave-equation MVA, we update the model based on differences between two images, one simulated in the background model and an enhanced version of this image. If the enhanced version of the image goes too far from the reference image, we run the risk of subtracting events corresponding to different reflectors. This phenomenon is usually referred-to in the literature as “cycle skipping” and various strategies have been designed to ameliorate this problem, e.g. by optimal selection of frequencies used for velocity analysis [Sirgue and Pratt 2004; Albertin 2006]. However, alternative methods used to evaluate image accuracy, e.g. differential semblance [Symes and Carazzone 1991; Shen et al. 2003], have the best potential to ameliorate this situation. In this case wave-equation MVA analyzes difference between image traces within common-image gathers which are likely to be similar enough from one-another such as to avoid the cycle-skipping problem. Even in this case, the assumption made is that the nearby traces in a gather are sampled well-enough, i.e. the seismic events differ by only a fraction of the wavelet, which is a function of image sampling and frequency content. In practice, there is no absolute guarantee that nearby events are closely related to one another, although this is more likely to be true for DSO than it is for direct image differencing.

In this paper, we concentrate on the implementation of the wave-equation migration velocity analysis operators for various wave-equation migration configurations. The main objective of the paper is to derive the linearized operators linking perturbations of the slowness model to the corresponding perturbations of the seismic wavefield and image. All our theoretical development is formulated under the single scattering (Born) approximation applied to acoustic waves. We begin by describing the MVA operators corresponding to zero-offset, survey-sinking and shot-record wave-equation migration frameworks. We describe the theoretical background for each operator and emphasize the similarities and differences between the different operators. Throughout the paper, we use pseudo-code to illustrate implementation strategies and data flow for the various wave-equation MVA operators. Finally, we illustrate the wave-equation MVA operators with impulse responses corresponding to simple and complex models. We leave outside the scope of this paper the procedure in which the discussed operators are used for migration velocity analysis.
WAVE-EQUATION MIGRATION AND VELOCITY ANALYSIS OPERATORS

The conceptual framework of wave-equation MVA is similar to that of conventional (ray-based) MVA in that the source of information for velocity updating is extracted from features of migrated images. This is in contrast with wave-equation tomography (or inversion), where the source of information is represented by the mismatch between recorded and simulated data. The main difference between wave-equation MVA and ray-based MVA is that the carrier of information from the migrated images to the velocity model is represented by the entire extrapolated wavefield and not by a rayfield constructed from selected points in the image based on an approximate velocity model.

The key element for the wave-equation MVA technique is a definition of an image perturbation corresponding to the difference between the image obtained with a known background velocity model and an improved image. Such image perturbations can be constructed using straight differences between images (Biondi and Sava, 1999; Albertin et al., 2006a), or by examining moveout parameters in migrated images (Sava and Biondi, 2004a,b; Shen et al., 2005; Albertin et al., 2006b; Maharramov and Albertin, 2007). Then, using wave-equation MVA operators, the image perturbations can be translated into slowness perturbations which update the model. The direct analogy between wave-based MVA and ray-based MVA is the following: wave-based methods use image perturbations and back-propagation using waves, while ray-based methods use traveltime perturbations and back-propagation using rays. Thus, wave-equation MVA benefits from all the characteristics of wave-based imaging techniques, e.g. stability in areas of large velocity variation, while remaining conceptually similar to conventional traveltime-based MVA.

We can formulate wave-equation migration and velocity analysis for different configurations in which we process the recorded data. There are two main classes of wave-equation migration, survey-sinking migration and shot-record migration (Claerbout, 1985), which differ in the way in which recorded data are processed. Both wave-equation imaging techniques use similar algorithms for downward continuation and, in theory, produce identical images for identical implementation of extrapolation operators and if all data are used for imaging (Berkhout, 1982; Biondi, 2003). The main difference is that shot-record migration is used to process separate seismic experiments (shots) sequentially, while survey-sinking migration is used to process all seismic experiments (shots) simultaneously. The shot-record operators are more computationally expensive but less memory intensive than the survey-sinking operators. A special case of survey-sinking migration assumes the sources and receivers are coincident on the acquisition surface, a technique usually described as the exploding reflector model (Loewenthal et al., 1976) applicable to zero-offset data. All operators described here can be used in models characterized by complex wave propagation (multipathing).

In all situations, wave-equation migration can be formulated as consisting of two main steps. The first step is wavefield reconstruction (abbreviated w.r. for the rest of
this paper) at all locations in space and using all frequencies from the recorded data as boundary conditions. This step requires numeric solutions to a form of wave equation, typically the one-way acoustic wave equation. The second step is the imaging condition (abbreviated i.c. for the rest of this paper), which is used to extract from the reconstructed wavefield(s) the locations where reflectors occur. This step requires numeric implementation of image processing techniques, e.g. cross-correlation, which evaluate properties of the wavefield indicating the presence of reflectors. Needless to say, the two steps are not implemented sequentially in practice, since the size of the wavefield is usually large and cannot be handled efficiently on conventional computers. Instead, wavefield reconstruction and imaging condition are implemented on-the-fly, avoiding expensive data storage and retrieval. Wave-equation MVA requires implementation of an additional procedure which links image and slowness perturbations. This link is given by a wavefield scattering operation (abbreviated w.s. for the rest of this paper) which is derived by linearization from conventional wavefield extrapolation operators.

In the following sections, we describe the migration and velocity analysis operators for the various imaging configurations. We begin with zero-offset imaging under the exploding reflector model, because this is the simplest wave-equation imaging framework and can aid our understanding of both survey-sinking and shot-record migration and velocity analysis frameworks. We then continue with a description of the wave-equation migration velocity analysis operator for multi-offset data using the survey-sinking and shot-record migration configuration. For each configuration, we describe the implementation of the forward operator (used to translate model perturbations into image perturbations) and of the adjoint operator (used to transform image perturbations into model perturbations). Both forward and adjoint operators are necessary for the implementation of efficient numeric conjugate gradient optimization [Claerbout, 1985]. Throughout this paper, we are using the following notations and naming conventions:

- $\omega$: angular frequency
- $z$: depth coordinate
- $m = \{m_x, m_y\}$: midpoint coordinates
- $h = \{h_x, h_y\}$: half-offset coordinates
- $k_m = \{k_{m_x}, k_{m_y}\}$: midpoint wavenumbers
- $k_h = \{k_{h_x}, k_{h_y}\}$: half-offset wavenumbers
- $s(m)$: medium slowness
- $s_0(m)$: background slowness
• $u(m)$ or $u(m, h)$: wavefield at frequency $\omega$ and depth $z$ for zero-offset and multiple-offset data, respectively
• $\Delta u(m)$ or $\Delta u(m, h)$: scattered wavefield at frequency $\omega$ and depth $z$ for zero-offset and multiple-offset data, respectively
• $r(m)$ or $r(m, h)$: image at depth $z$ for zero-offset and multiple-offset data, respectively
• i.c.: imaging condition
• w.r.: wavefield extrapolation
• w.s.: wavefield scattering
• $\mathcal{E}^\pm [:]$: extrapolation operator (causal for $+$, anticausal for $-$)
• $\mathcal{F}^\pm [:]$: forward scattering operator (causal for $+$, anticausal for $-$)
• $\mathcal{A}^\mp [:]$: adjoint scattering operator (causal for $+$, anticausal for $-$)

**Zero-offset migration and velocity analysis**

Wavefield reconstruction for zero-offset migration based on the one-way wave-equation is performed by recursive phase-shift in depth starting with data recorded on the surface as boundary conditions. In this configuration, the imaging condition extracts the image as time $t = 0$ from the reconstructed wavefield at every location in space. Thus, the surface data need to be extrapolated backward in time which is achieved by selecting the appropriate sign of the phase-shift operation (which depends on the sign convention for temporal Fourier transforms):

$$ u_{z+\Delta z}(m) = e^{-ik_z\Delta z} u_z(m). $$

In equation 1, $u_z(m)$ represents the acoustic wavefield at depth $z$ for a given frequency $\omega$ at all positions in space $m$, and $u_{z+\Delta z}(m)$ represents the same wavefield extrapolated to depth $z + \Delta z$. The phase shift operation uses the depth wavenumber $k_z$ which is defined by the single square-root (SSR) equation

$$ k_z = \sqrt{[2\omega s(m)]^2 - |k_m|^2}, $$

where $s(m)$ represents the spatially-variable slowness at depth level $z$. Equations 1–2 describe wavefield extrapolation using a pseudo-differential operator, which justifies our use of laterally-varying slowness $s(m)$. As indicated earlier, the image is obtained from this extrapolated wavefield by selection of time $t = 0$, which is typically implemented as summation of the extrapolated wavefield over frequencies:

$$ r_z(m) = \sum_\omega u_z(m, \omega). $$

(3)
Phase-shift extrapolation using wavenumbers computed using equations 1 and 2 is not feasible in media with lateral variation. Instead, implementation of such operators is done using approximations implemented in a mixed space-wavenumber domain (Stoffa et al., 1990; Ristow and Ruhl, 1994; Huang et al., 1999). A brief summary the mixed-domain implementation of the split-step Fourier (SSF) operator is presented in Appendix A.

For velocity analysis, we assume that we can separate the total slowness $s(m)$ into a known background component $s_0(m)$ and an unknown component $\Delta s(m)$. With this convention, we can linearize the depth wavenumber $k_z$ relative to the background slowness $s_0$ using a truncated Taylor series expansion

$$k_z \approx k_{z0} + \frac{dk_z}{ds} \mid_{s_0} \Delta s(m) , \quad (4)$$

where the depth wavenumber in the background medium characterized by slowness $s_0(m)$ is

$$k_{z0} = \sqrt{[2\omega s_0(m)]^2 - |k_m|^2} . \quad (5)$$

Here, $s_0(m)$ represents the spatially-variable background slowness at depth level $z$. Using the wavenumber linearization from equation 4, we can reconstruct the acoustic wavefields in the background model using a phase-shift operation

$$u_{z+\Delta z}(m) = e^{-ik_{z0}\Delta z}u_z(m) . \quad (6)$$

We can represent wavefield extrapolation using a generic solution to the one-way wave-equation using the notation $u_{z+\Delta z}(m) = E_{ZOM}[2s_{0z}(m), u_z(m)]$. This notation indicates that the wavefield $u_{z+\Delta z}(m)$ is reconstructed from the wavefield $u_z(m)$ using the background slowness $s_0(m)$. This operation is repeated independently for all frequencies $\omega$. A typical implementation of zero-offset wave-equation migration uses the following algorithm:

**ZERO-OFFSET MIGRATION ALGORITHM**

```plaintext
@ $\omega$
    read $u(m)$
    $z = z_{min} \ldots z_{max}$
@ $z$ and $\omega$ (optional)
    write $u(m)$
@ $z$
    read $r(m)$
    I.C.
    $r(m) + = u(m)$
@ $z$
    write $r(m)$
W.R.
    $u(m) = E_{ZOM}[2s_0(m), u(m)]$
```

```plaintext
}
```
A seismic image is produced using migration by wavefield extrapolation as follows: for each frequency, read data at all spatial locations \( m \); then, for each depth, sum the wavefield into the image at that depth level (i.e. apply the imaging condition) and then reconstruct the wavefield to the next depth level (i.e. perform wavefield extrapolation). The “-” sign in this algorithm indicates that extrapolation is anti-causal (backward in time), and the factor “2” indicates that we are imaging data recorded in two-way traveltime with an algorithm designed under the exploding reflector model. Wavefield extrapolation is usually implemented in a mixed domain (space-wavenumber), as briefly summarized in Appendix A.

The wavefield perturbation \( \Delta u(m) \) caused at depth \( z + \Delta z \) by a slowness perturbation \( \Delta s(m) \) at depth \( z \) is obtained by subtraction of the wavefields extrapolated from \( z \) to \( z + \Delta z \) in the true and background models:

\[
\Delta u_{z + \Delta z}(m) = e^{-ik_z\Delta z}u_z(m) - e^{-ik_0\Delta z}u(z, m) \\
= e^{-ik_0\Delta z} \left[ e^{-i\frac{dk}{ds}|_{s_0} \Delta s(m) \Delta z} - 1 \right] u_z(m).
\]  

(7)

Here, \( \Delta u(m) \) and \( u(m) \) correspond to a given depth level \( z \) and frequency \( \omega \). A similar relation can be applied at all depths and all frequencies.

Equation (7) establishes a non-linear relation between the wavefield perturbation \( \Delta u(m) \) and the slowness perturbation \( \Delta s(m) \). Given the complexity and cost of numeric optimization based on non-linear relations between model and wavefield parameters, it is desirable to simplify this relation to a linear relation between model and data parameters. Assuming small slowness perturbations, i.e. small phase perturbations, the exponential function \( e^{\pm i\frac{dk}{ds}|_{s_0} \Delta s(m) \Delta z} \) can be linearized using the approximation \( e^{i\phi} \approx 1 + i\phi \) which is valid for small values of the phase \( \phi \). Therefore the wavefield perturbation \( \Delta u(m) \) at depth \( z \) can be written as

\[
\Delta u(m) \approx -i \frac{dk}{ds}|_{s_0} \Delta z \, u(m) \, \Delta s(m).
\]

(8)

Equation (8) defines the zero-offset forward scattering operator \( \mathcal{F}_{ZOM}[u(m), 2s_0(m), \Delta s(m)] \), producing the scattered wavefield \( \Delta u(m) \) from the slowness perturbation \( \Delta s(m) \), based on the background slowness \( s_0(m) \) and background wavefield \( u(m) \) at a given frequency \( \omega \). The image perturbation at depth \( z \) is obtained from the scattered wavefield using the time \( t = 0 \) imaging condition, similar to the procedure used for imaging in the background medium:

\[
\Delta r(m) = \sum_{\omega} \Delta u(m, \omega).
\]

(9)
Given an image perturbation $\Delta r (m)$, we can construct the scattered wavefield by the adjoint of the imaging condition

$$\Delta u (m, \omega) = \Delta r (m),$$

(10)

for every frequency $\omega$. Then, the slowness perturbation at depth $z$ and frequency $\omega$ caused by a wavefield perturbation at depth $z$ under the influence of the background wavefield at the same depth $z$ can be written as

$$\Delta s (m) \approx + i \frac{dk_z}{ds} \left| s_0 \right| \Delta u (m) \Delta u (m) = + i \Delta z \frac{2 \omega u (m) \Delta u (m)}{1 - k_m^2 \frac{\Delta s (m)}{2 \omega s (m)}}.$$

(11)

Equation [11] defines the adjoint scattering operator $A_{ZOM}^* [u (m), 2s_0 (m), \Delta u (m)]$, producing the slowness perturbation $\Delta s (m)$ from the scattered wavefield $\Delta u (m)$, based on the background slowness $s_0 (m)$ and background wavefield $u (m)$. A typical implementation of zero-offset forward and adjoint scattering uses the following algorithms:

**ZERO-OFFSET FORWARD SCATTERING ALGORITHM**

@ $\omega$
initialize $\Delta u (m) = 0$

@ $z$ and $\omega$
read $u (m)$

w.s.
$\Delta u (m) + = F_{ZOM} [u (m), 2s_0 (m), \Delta u (m)]$

@ $z$
read $\Delta r (m)$

l.c.
$\Delta r (m) + = \Delta u (m)$

@ $z$
write $\Delta r (m)$

w.r.
$\Delta u (m) = E_{ZOM} [2s_0 (m), \Delta u (m)]$

**ZERO-OFFSET ADJOINT SCATTERING ALGORITHM**

@ $\omega$
initialize $\Delta u (m) = 0$

@ $z$ and $\omega$
read $u (m)$

w.r.
$\Delta u (m) = E_{ZOM}^* [2s_0 (m), \Delta u (m)]$
The forward zero-offset wave-equation MVA operator follows a similar pattern to the implementation of the downward continuation operator: for each frequency and for each depth, read the slowness perturbation $\Delta s$ at all spatial locations $m$, then apply the scattering operator (w.s.) given equation [11] to the slowness perturbation and accumulate the scattered wavefield for downward continuation; then, apply the imaging condition (i.c.) producing the image perturbation $\Delta r$ at depth $z$ and then reconstruct the scattered wavefield backward in time using the wavefield extrapolation operator (w.r.) to the next depth level. The adjoint zero-offset wave-equation MVA operator also follows a similar pattern to the implementation of the downward continuation operator: for each frequency and for each depth, reconstruct the scattered wavefield forward in time using the wavefield extrapolation operator (w.r.) to the next depth level, then apply the adjoint of the imaging condition (i.c.) by adding the image to the scattered wavefield and then apply the adjoint wavefield scattering operator (w.s.) to obtain the slowness perturbation $\Delta s$. Both forward and adjoint scattering algorithms require the background wavefield, $u$, to be precomputed at all depth levels, although more efficient implementations using optimal checkpointing are possible (Symes, 2007). Scattering and wavefield extrapolation are implemented in the mixed space-wavenumber domain, as briefly explained in Appendix A.

Survey-sinking migration and velocity analysis

Wavefield reconstruction for multi-offset migration based on the one-way wave-equation under the survey-sinking framework (Claerbout, 1985) is implemented similarly to the zero-offset case by recursive phase-shift of prestack wavefields

$$u_{z+\Delta z}(m, h) = e^{-ik_z\Delta z}u_z(m, h)$$

(12)

followed by extraction of the image at time $t = 0$. Here, $m$ and $h$ stand for midpoint and half-offset coordinates, respectively, defined according to the relations

$$m = \frac{r + s}{2}$$

(13)

$$h = \frac{r - s}{2}$$

(14)

where $s$ and $r$ are coordinates of sources and receivers on the acquisition surface. In equation [12] $u_z(m, h)$ represents the acoustic wavefield for a given frequency $\omega$ at
all midpoint positions \( \mathbf{m} \) and half-offsets \( \mathbf{h} \) at depth \( z \), and \( u_{z+\Delta z} (\mathbf{m}, \mathbf{h}) \) represents the same wavefield extrapolated to depth \( z + \Delta z \). The phase shift operation uses the depth wavenumber \( k_z \) which is defined by the double square-root (DSR) equation

\[
k_z = \sqrt{[\omega s (\mathbf{m} - \mathbf{h})]^2 - \frac{|k_m - k_h|^2}{2}} + \sqrt{[\omega s (\mathbf{m} + \mathbf{h})]^2 - \frac{|k_m + k_h|^2}{2}}. \tag{15}
\]

The image is obtained from this extrapolated wavefield by selection of time \( t = 0 \), which is usually implemented as summation over frequencies:

\[
r_z (\mathbf{m}, \mathbf{h}) = \sum_{\omega} u_z (\mathbf{m}, \mathbf{h}, \omega). \tag{16}
\]

Similarly to the derivation done in the zero-offset case, we can assume the separation of the extrapolation slowness \( s (\mathbf{m}) \) into a background component \( s_0 (\mathbf{m}) \) and an unknown perturbation component \( \Delta s (\mathbf{m}) \). Then we can construct a wavefield perturbation \( \Delta u (\mathbf{m}, \mathbf{h}) \) at depth \( z \) and frequency \( \omega \) related linearly to the slowness perturbation \( \Delta s (\mathbf{m}) \). Linearizing the depth wavenumber given by the DSR equation [15] relative to the background slowness \( s_0 (\mathbf{m}) \), we obtain

\[
k_z \approx k_{z0} + \frac{dk_z}{ds} s_0 (\mathbf{m} - \mathbf{h}) + \frac{dk_{zr}}{ds} s_0 (\mathbf{m} + \mathbf{h}) \Delta s (\mathbf{m} + \mathbf{h}), \tag{17}
\]

where the depth wavenumber in the background medium is

\[
k_{z0} = \sqrt{[\omega s_0 (\mathbf{m} - \mathbf{h})]^2 - \frac{|k_m - k_h|^2}{2}} + \sqrt{[\omega s_0 (\mathbf{m} + \mathbf{h})]^2 - \frac{|k_m + k_h|^2}{2}}. \tag{18}
\]

Here, \( s_0 (\mathbf{m}) \) represents the spatially-variable background slowness at depth level \( z \). Using the wavenumber linearization given by equation [17] we can reconstruct the acoustic wavefields in the background model using a phase-shift operation

\[
u_{z+\Delta z} (\mathbf{m}, \mathbf{h}) = e^{-ik_{z0}\Delta z} u_z (\mathbf{m}, \mathbf{h}). \tag{19}
\]

We can represent wavefield extrapolation using a generic solution to the one-way wave-equation using the notation \( u_{z+\Delta z} (\mathbf{m}, \mathbf{h}) = E_{SSM}^{-}[s_0 (\mathbf{m}), u_z (\mathbf{m}, \mathbf{h})] \). This notation indicates that the wavefield \( u_{z+\Delta z} (\mathbf{m}, \mathbf{h}) \) is reconstructed from the wavefield \( u_z (\mathbf{m}, \mathbf{h}) \) using the background slowness \( s_0 (\mathbf{m}) \). This operation is repeated independently for all frequencies \( \omega \). A typical implementation of survey-sinking wave-equation migration uses the following algorithm:

**SURVEY-SINKING MIGRATION ALGORITHM**
\[ \omega = \omega_{\text{min}} \ldots \omega_{\text{max}} \{ \]
\[ \text{read } u(m, h) \]
\[ z = z_{\text{min}} \ldots z_{\text{max}} \{ \]
\[ \text{write } u(m, h) \]
\[ \text{I.C. } r(m, h) + = u(m, h) \]
\[ \text{write } r(m, h) \]
\[ \text{W.R. } u(m, h) = E_{\text{SSM}}^{-}[s_0(m), u(m, h)] \]
\]

This algorithm is similar to the one described in the preceding section for zero-offset migration, except that the wavefield and image are parametrized by midpoint and half-offset coordinates and that the depth wavenumber used in the extrapolation operator is given by the DSR equation using the background slowness \( s_0(m) \). Wavefield extrapolation is usually implemented in a mixed domain (space-wavenumber), as briefly summarized in Appendix A.

Similarly to the derivation of the wavefield perturbation in the zero-offset migration case, we can write the linearized wavefield perturbation for survey-sinking migration as

\[
\Delta u(m, h) \approx -i \left. \frac{dk_z a}{ds} \right| _{s_0} \Delta s(m - h) \Delta z u(m, h) \\
- i \left. \frac{dk_z r}{ds} \right| _{s_0} \Delta s(m + h) \Delta z u(m, h) \\
\approx -i \Delta z \frac{\omega u(m, h) \Delta s(m - h)}{\sqrt{1 - \left[ \frac{|k_m - k_h|}{2 \omega s_0(m - h)} \right]^2}} \\
- i \Delta z \frac{\omega u(m, h) \Delta s(m + h)}{\sqrt{1 - \left[ \frac{|k_m + k_h|}{2 \omega s_0(m + h)} \right]^2}} . \tag{20}
\]

Equation 20 defines the forward scattering operator \( F_{\text{SSM}}^{-}[u(m, h), s_0(m), \Delta s(m, h)] \), producing the scattered wavefield \( \Delta u(m, h) \) from the slowness perturbation \( \Delta s(m) \), based on the background slowness \( s_0(m) \) and background wavefield \( u(m, h) \). The image perturbation at depth \( z \) is obtained from the scattered wavefield using the time \( t = 0 \) imaging condition, similar to the procedure used for imaging in the background medium:

\[
\Delta r(m, h) = \sum_\omega \Delta u(m, h, \omega) . \tag{21}
\]

Given an image perturbation \( \Delta r(m, h) \), we can construct the scattered wavefield by the adjoint of the imaging condition

\[
\Delta u(m, h, \omega) = \Delta r(m, h) , \tag{22}
\]
for every frequency $\omega$. Then, similarly to the procedure used in the zero-offset case, the slowness perturbation at depth $z$ caused by a wavefield perturbation at depth $z$ under the influence of the background wavefield at the same depth $z$ can be written as

$$
\Delta s (m-h) \approx + i \frac{dk_{zs}}{ds} \Delta z \frac{\omega u(m,h)\Delta u(m,h)}{1 - \left[ \frac{|k_m-k_h|}{2\omega s_0(m-h)} \right]^2},
$$

and

$$
\Delta s (m+h) \approx + i \frac{dk_{zr}}{ds} \Delta z \frac{\omega u(m,h)\Delta u(m,h)}{1 - \left[ \frac{|k_m+k_h|}{2\omega s_0(m+h)} \right]^2}. \tag{24}
$$

Equations 23-24 define the adjoint scattering operator $A_{SSM}^+ [u(m,h), s_0(m), \Delta u(m,h)]$, producing the slowness perturbation $\Delta s (m)$ from the scattered wavefield $\Delta u (m,h)$, based on the background slowness $s_0(m)$ and background wavefield $u(m,h)$. A typical implementation of survey-sinking forward and adjoint scattering follows the algorithms:

---

**SURVEY-SINKING FORWARD SCATTERING ALGORITHM**

\[ \omega = \omega_{\text{min}} \ldots \omega_{\text{max}} \{
\]

\[ @ \omega \quad \text{initialize } \Delta u (m, h) = 0 \]

\[ @ z \text{ and } \omega \quad \text{read } u (m, h) \]

\[ @ z \quad \text{read } \Delta s (m) \]

\[ \text{W.S. (source)} \quad \Delta u (m, h) = \mathcal{F}_{SSM}^- [u(m,h), s_0(m-h), \Delta s (m-h)] \]

\[ \text{W.S. (receiver)} \quad \Delta u (m, h) = \mathcal{F}_{SSM}^- [u(m,h), s_0(m+h), \Delta s (m+h)] \]

\[ @ z \quad \text{read } \Delta r (m, h) \]

\[ \text{I.C.} \quad \Delta r (m, h) = \Delta u (m, h) \]

\[ @ z \quad \text{write } \Delta r (m, h) \]

\[ \text{W.R.} \quad \Delta u (m, h) = \mathcal{E}_{SSM}^- [s_0(m), \Delta u (m, h)] \]

\[ \} \]

---

**SURVEY-SINKING ADJOINT SCATTERING ALGORITHM**
\[ \omega = \omega_{\text{min}} \ldots \omega_{\text{max}} \{ \]

\[ @ \omega \]

initialize \( \Delta u (m, h) = 0 \)

\[ @ z \text{ and } \omega \]

\[ \Delta u (m, h) = E_{SSM}^+ [s_0 (m), \Delta u (m, h)] \]

\[ w.r. @ z \]

read \( u (m, h) \)

\[ @ z \]

\[ \Delta r (m, h) \]

\[ l.c. @ z \]

\[ \Delta u (m, h) + = \Delta r (m, h) \]

\[ @ z \text{ w.s. (source)} \]

\[ \Delta s (m - h) + = A_{SSM}^{-} [u (m, h), s_0 (m - h), \Delta u (m, h)] \]

\[ @ z \text{ w.s. (receiver)} \]

\[ \Delta s (m + h) + = A_{SSM}^{-} [u (m, h), s_0 (m + h), \Delta u (m, h)] \]

\[ @ z \]

write \( \Delta s (m) \)

These algorithms are similar to the ones described in the preceding section for zero-offset migration, except that the wavefield and image are parametrized by midpoint and half-offset coordinates. Furthermore, the two square-roots corresponding to the DSR equation update the slowness model separately, thus characterizing the source and receiver propagation paths to the image positions. Both forward and adjoint scattering algorithms require the background wavefield, \( u (m, h) \), to be precomputed at all depth levels. Scattering and wavefield extrapolation are implemented in the mixed space-wavenumber domain, as briefly explained in Appendix A.

**Shot-record migration and velocity analysis**

Wavefield reconstruction for multi-offset migration based on the one-way wave-equation under the shot-record framework is performed by separate recursive extrapolation of the source and receiver wavefields, \( u_s \) and \( u_r \). The wavefield extrapolation progresses forward in time (causal) for the source wavefield and backward in time (anti-causal) for the receiver wavefield:

\[ u_{sz + \Delta z} (m) = e^{ik_{z}\Delta z} u_{sz} (m) \quad (25) \]

\[ u_{rz + \Delta z} (m) = e^{-ik_{z}\Delta z} u_{rz} (m) \quad (26) \]

In equations \( u_{sz} (m) \) and \( u_{rz} (m) \) represent the source and receiver acoustic wavefield for a given frequency \( \omega \) at all positions in space \( m \) at depth \( z \), and \( u_{sz + \Delta z} (m) \) and \( u_{rz + \Delta z} (m) \) represent the same wavefields extrapolated to depth \( z + \Delta z \). The phase shift operation uses the depth wavenumber \( k_z \) which is defined by the single square-root (SSR) equation

\[ k_z = \sqrt{[\omega s (m)]^2 - |k_m|^2} \quad (27) \]

The image is obtained from the extrapolated wavefields by selection of the zero cross-correlation lags in space of time between the source and receiver wavefields, an oper-
lation which is usually implemented as summation over frequencies:

\[ r_z(m) = \sum_\omega u_{sz}(m,\omega)u_{rz}(m,\omega). \]  

(28)

An alternative imaging condition ([Sava and Fomel, 2006]) preserves the space and time cross-correlation lags in the image.

Linearizing the depth wavenumber given by the equation 27 relative to the background slowness \( s_0(m) \) similarly to the case case of zero-offset migration, we can reconstruct the acoustic wavefields in the background model using a phase-shift operation

\[ u_{sz+\Delta z}(m) = e^{+ik_0\Delta z}u_{sz}(m), \quad (29) \]
\[ u_{rz+\Delta z}(m) = e^{-ik_0\Delta z}u_{rz}(m), \quad (30) \]

which define the causal \( \mathcal{E}_{SRM}^+[s_0z(m),u_z(m)] \) and the anti-causal \( \mathcal{E}_{SRM}^-[s_0z(m),u_z(m)] \) wavefield extrapolation operators for shot-record migration constructed using the background slowness \( s_0(m) \) and producing the wavefields \( u_{sz+\Delta z}(m) \) and \( u_{rz+\Delta z}(m) \) at depth \( z + \Delta z \) from the wavefields \( u_{sz}(m) \) and \( u_{rz}(m) \) at depth \( z \), respectively. A typical implementation of shot-record wave-equation migration follows the algorithm:

**SHOT-RECORD MIGRATION ALGORITHM**

\[ \omega = \omega_{\min} \ldots \omega_{\max} \{ \]
\[ @ \omega \quad \{ \]
\[ \begin{align*}
\text{read } u_s(m) \text{ and } u_r(m) \\
\text{z = z}_{\min} \ldots z_{\max} \{ \]
\[ @ z \quad \{ \]
\[ \begin{align*}
\text{write } u_s(m) \text{ and } u_r(m) \\
\text{I.C. } \quad \{ \]
\[ r(m) + = \overline{u_s(m)u_r(m)} \]
\[ @ z \quad \{ \]
\[ \text{write } r(m) \]
\[ \text{W.R. } \quad \{ \]
\[ u_s(m) = \mathcal{E}_{SRM}^+[s_0(m),u_s(m)] \]
\[ \text{W.R. } \quad \{ \]
\[ u_r(m) = \mathcal{E}_{SRM}^-[s_0(m),u_r(m)] \]
\[ \} \]

This algorithm is similar to the one used for zero-offset or survey sinking migration, except that the source and receiver wavefields are reconstructed separately using wavefield extrapolation. Unlike the zero-offset extrapolation operator, the shot-record extrapolation operator uses the background slowness \( s_0 \) since the operation involves sinking of the source and receiver wavefields from the surface toward the image positions. Wavefield extrapolation is usually implemented in a mixed domain (space-wavenumber), as briefly summarized in Appendix A.
Similarly to the derivation of the wavefield perturbation in the zero-offset migration case, we can write the linearized wavefield perturbation for shot-record migration as

$$\Delta u_s (m) \approx +i \frac{dk_z}{ds} \Delta z u_s (m) \Delta s (m)$$

$$\approx +i \Delta z \frac{\omega u_s (m) \Delta s (m)}{\sqrt{1 - \left[ \frac{|k_m|}{\omega s_0 (m)} \right]^2}},$$

(31)

and

$$\Delta u_r (m) \approx -i \frac{dk_z}{ds} \Delta z u_r (m) \Delta s (m)$$

$$\approx -i \Delta z \frac{\omega u_r (m) \Delta s (m)}{\sqrt{1 - \left[ \frac{|k_m|}{\omega s_0 (m)} \right]^2}},$$

(32)

Equations 31-32 define the forward scattering operators $F_{SRM}^\pm [u (m), s_0 (m), \Delta s (m)]$ producing the scattered wavefields $\Delta u (m)$ from the slowness perturbation $\Delta s (m)$, based on the background slowness $s_0 (m)$ and background wavefield $u (m)$. In this case, the symbol $u$ stands for either $u_s$ or $u_r$, given the appropriate choice of sign in the forward scattering operator. The image perturbation at depth $z$ is obtained from the source and receiver scattered wavefields using the relation

$$\Delta r (m) = \sum_\omega \left( u_s (m, \omega) \Delta u_r (m, \omega) + \overline{\Delta u_s (m, \omega) u_r (m, \omega)} \right),$$

(33)

which corresponds to the frequency-domain zero-lag cross-correlation of the source and receiver wavefields required by the imaging condition.

Given an image perturbation $\Delta r$, we can construct the scattered source and receiver wavefields by the adjoint of the imaging condition

$$\Delta u_s (m) = u_r (m) \overline{\Delta r (m)},$$

(34)

$$\Delta u_r (m) = u_s (m) \Delta r (m),$$

(35)

for every frequency $\omega$. Then, the slowness perturbations due to the source and receiver wavefields at depth $z$ under the influence of the background source and receiver wavefields at the same depth $z$ can be written as

$$\Delta s_s (m) \approx -i \frac{dk_z}{ds} \Delta z u_s (m) \Delta u_s (m)$$

$$\approx -i \Delta z \frac{\omega u_s (m) \Delta u_s (m)}{\sqrt{1 - \left[ \frac{|k_m|}{\omega s_0 (m)} \right]^2}},$$

(36)
and

\[
\Delta s_r (m) \approx -i \frac{dk_z}{ds} |s_0| \Delta z \frac{u_r (m) \Delta u_r (m)}{|s_0|} \approx -i \Delta z \frac{\omega u_r (m) \Delta u_r (m)}{\sqrt{1 - \left| \frac{k_m}{\omega s_0 (m)} \right|^2}}.
\]

Equations 36 and 37 define the adjoint scattering operators \( A_{SRM}^\pm [u (m) , s_0 (m) , \Delta u (m)] \), producing the slowness perturbation \( \Delta s (m) \) from the scattered wavefield \( \Delta u (m) \), based on the background slowness \( s_0 (m) \) and background wavefield \( u (m) \). In this case, \( u \) stands for either \( u_s \) or \( u_r \), given the appropriate choice of sign in the adjoint scattering operator. A typical implementation of shot-record forward and adjoint scattering follows the algorithms:

---

**SHOT-RECORD FORWARD SCATTERING ALGORITHM**

\[
\omega = \omega_{min} \ldots \omega_{max}\{ \\
\text{initialize } \Delta u_s (m) = 0 \text{ and } \Delta u_r (m) = 0 \\
\text{read } u_s (m) \text{ and } u_r (m) \\
\text{read } \Delta s (m) \\
\text{read } \Delta r (m) \\
\text{read } \Delta r (m) + = u_s (m) \Delta u_r (m) \\
\text{write } \Delta r (m) \\
\text{read } \Delta u_s (m) = \mathcal{E}_{SRM}^+ [s_0 (m) , \Delta u_s (m)] \\
\text{write } \Delta u_r (m) = \mathcal{E}_{SRM}^- [s_0 (m) , \Delta u_r (m)] \\
\}
\]

---

**SHOT-RECORD ADJOUT SCATTERING ALGORITHM**

\[
\omega = \omega_{min} \ldots \omega_{max}\{ \\
\text{initialize } \Delta u_s (m) = 0 \text{ and } \Delta u_r (m) = 0 \\
\text{read } u_s (m) \text{ and } u_r (m) \\
\text{read } \Delta r (m) \\
\text{read } \Delta u_r (m) = \mathcal{E}_{SRM}^+ [s_0 (m) , \Delta u_r (m)] \\
\text{read } \Delta u_s (m) = \mathcal{E}_{SRM}^- [s_0 (m) , \Delta u_s (m)] \\
\text{read } \Delta u_r (m) + = u_s (m) \Delta r (m) \\
\}
\]
\[ \Delta u_s(m) + = u_r(m) \Delta r(m) \]

\[ \Delta s(m) + = A_{SRM}^{+} [u_r(m), s_0(m), \Delta u_r(m)] \]

\[ \Delta s(m) + = A_{SRM}^{-} [u_s(m), s_0(m), \Delta u_s(m)] \]

These algorithms are similar to the one used for zero-offset or survey sinking migration, except that the source and receiver wavefields are reconstructed separately using wavefield extrapolation. Unlike the zero-offset scattering operators, the shot-record scattering operators use the background slowness \( s_0 \) since the operation involves sinking of the source and receiver wavefields from the surface toward the image positions. Both forward and adjoint scattering algorithms require the background wavefields, \( u_s(m) \) and \( u_r(m) \), to be precomputed at all depth levels. Scattering and wavefield extrapolation are implemented in the mixed space-wavenumber domain, as briefly explained in Appendix A.

**SUMMARY OF OPERATORS**

All wave-equation migration velocity analysis operators described in the preceding sections are similar in that they relate perturbations of the image with perturbations of the (slowness) model. In all cases, this velocity estimation procedure takes advantage of features of migrated images which indicate incorrect imaging. The imaging inaccuracies can have several causes, i.e. incorrect downward continuation, irregular illumination, limited acquisition aperture etc., but the wave-equation MVA operators translate all inaccuracies in model updates. This feature, however, is a fundamental limitation of all migration velocity analysis techniques and we do not expand on this topic further.

Since the migration velocity analysis operators link image perturbations with slowness perturbations, they are all composed of several common parts, but with implementations that are specific for each imaging configuration. Thus, a wave-equation MVA operator is composed of an extrapolation operator (for wavefield reconstruction from recorded data), an imaging operator (for image construction from reconstructed wavefields) and a scattering operator (for relating wavefield perturbations to slowness perturbations). The following table summarizes the wave-equation MVA operator components in different imaging configurations, as described in detail in the preceding sections.
<table>
<thead>
<tr>
<th></th>
<th>extrapolation operator</th>
<th>imaging operator</th>
<th>scattering operator</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>equation</td>
<td>type</td>
<td>equation</td>
</tr>
<tr>
<td>zero-offset</td>
<td>SSR half backward</td>
<td>zero time</td>
<td>linearized SSR</td>
</tr>
<tr>
<td></td>
<td>equations 1, 2</td>
<td>equation 3</td>
<td>equations 8, 11</td>
</tr>
<tr>
<td>survey-sinking</td>
<td>DSR full backward</td>
<td>zero time</td>
<td>linearized DSR</td>
</tr>
<tr>
<td></td>
<td>equations 12, 15</td>
<td>equation 16</td>
<td>equations 20, 23, 24</td>
</tr>
<tr>
<td>shot-record</td>
<td>SSR full forward backward</td>
<td>zero time</td>
<td>linearized SSR</td>
</tr>
<tr>
<td></td>
<td>equations 25, 26, 27</td>
<td>equation 28</td>
<td>equations 31, 32, 36, 37</td>
</tr>
</tbody>
</table>

Figure 1: Schematic representation of the forward and adjoint operators for ray-based MVA and wave-based MVA. The forward operator $F$ applied to a slowness anomaly $\Delta s$ generates a traveltime perturbation (a) or an image perturbation (b). The ray-based adjoint MVA operator $A$ applied to the traveltime perturbation generates a slowness perturbation uniformly distributed along a ray normal to the reflector (c). The wave-based adjoint MVA operator $A$ applied to the image perturbation generates a slowness perturbation with a wider space distribution but with a relative focus at the location of the original slowness anomaly (d).
EXAMPLES

We illustrate the wave-equation migration velocity analysis operators using impulse responses corresponding to different imaging configurations. We concentrate on imaging in the zero-offset and shot-record frameworks, since they also implicitly characterize the essential elements of the survey-sinking framework. In all cases, we use wavefield reconstruction based on one-way wavefield extrapolation with the multi-reference split-step Fourier method (Stoffa et al., 1990; Popovici, 1996).

![Figure 2: Simple synthetic model with (a) linear $v(z)$ velocity and (b) a horizontal reflector.](geo2008NumericWEMVAoperators/flatWEMVA_vel.img)

A fundamental question concerning the wavefield scattering operator (w.s.) is what is its sensitivity for a given perturbation of the image or of the slowness model. This sensitivity is usually characterized using the so-called “sensitivity kernels” which are often discussed in the literature in the context of tomography problems. For wave-equation MVA, this topic was discussed in the context of zero-offset imaging by Sava and Biondi (2004a,b). The important topic of sensitivity and model resolution falls outside the scope of this paper, so we do not discuss it here in any detail. We merely concern ourselves with describing the behavior of the wave-equation MVA operators described earlier.

We can analyze the sensitivity of the wavefield scattering operator in two ways. The first option is to assume a localized slowness perturbation, compute image perturbations using the forward scattering operator and then return to the slowness perturbation using the adjoint scattering operator. The second option is to assume a localized image perturbation, compute the slowness perturbation using the adjoint scattering operator and then return to the image perturbation using the forward scattering operator.
As discussed in the preceding sections, the main difference between ray-based and wave-based MVA techniques is that the connection between measurements on the image and updates to the model is done with rays and waves, respectively. The impact of this fundamental difference is best seen if we analyze impulse responses of the wave-equation MVA and compare them with those of conventional traveltime tomography. Figure 1 shows a one-to-one comparison between the forward and adjoint operators for ray-based MVA (traveltime tomography) on the left and wave-based MVA on the right in the context of zero-offset imaging. Assuming a small slowness perturbation $\Delta s$, we can construct using the forward MVA operators a traveltime perturbation and an image perturbation corresponding to ray-based MVA (a) and wave-based MVA (b), respectively. For this zero-offset configuration, the ray-based MVA produces a traveltime anomaly strictly located on the reflector under the slowness anomaly, while the wave-based MVA produces an image anomaly distributed in space in the vicinity of the reflector. Then, we can construct respective slowness updates if we apply the ray-based and wave-based adjoint MVA operators to the traveltime perturbation and image perturbation, respectively. For the ray-based MVA, the slowness update spreads uniformly along a ray orthogonal to the reflector (c), while for wave-based MVA, the slowness update is distributed in space from the image perturbation to the surface, but with a concentration at the location of the true anomaly (d). Similar behavior characterizes wave-equation MVA under shot-record or survey-sinking frameworks.

The first set of examples corresponds to a simple model consisting of a linear $v(z)$
Figure 4: (a) Simulated zero-offset data and (b) simulated shot-record data for the model depicted in Figures 2(a), 2(b) with a source located at coordinates $x = 6$ km and $z = 0$ km.
Figure 5: (a) Zero-offset image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from Figure 3(a) and (b) zero-offset slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from panel (a).

Figure 6: (a) Shot-record image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from Figure 3(a) and (b) shot-record slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from panel (a).
Figure 7: (a) Zero-offset slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from Figure 3(b) and (b) zero-offset image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from panel (a).

geo2008NumericWEMVAoperators/flatWEMVA ZAdi4,ZFAdi4

Figure 8: (a) Shot-record slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from Figure 3(a) and (b) shot-record image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from panel (a).

geo2008NumericWEMVAoperators/flatWEMVA SAdi4,SFAdi4
velocity model and a horizontal reflector, Figures 2(a)-2(b). The velocity is linearly increasing from 1.5 km/s to 2.75 km/s. We simulate zero-offset data, Figure 4(a) and one shot corresponding to horizontal position $x = 6$ km, Figure 4(b).

Assuming a localized slowness perturbation, Figure 3(a), we can compute image perturbations using the forward scattering operators, as defined in the preceding sections. Figure 5(a) shows the image perturbation for the zero-offset case and Figure 6(a) shows the similar image perturbation for the shot-record case. As illustrated in Figure 1, the image perturbations are distributed in the vicinity of the reflector. Two interfering events are seen for the shot-record case, corresponding to the source and receiver wavefields, respectively.

Similarly, we can compute slowness perturbations using the adjoint scattering operators. Figure 5(b) shows the slowness perturbation for the zero-offset case computed from the image perturbation in Figure 5(a) and Figure 6(b) shows the similar slowness perturbation for the shot-record case computed from the image perturbation in Figure 6(a). As illustrated in Figure 1, the slowness perturbations are distributed in an area connecting the reflector to the surface, but with a relative focus at the location of the original anomaly. For the shot-record case, the back-projection splits toward the source and receivers, corresponding to the upward continuation of the source and receiver wavefields.

We can also analyze the wave-equation MVA operator sensitivity in another way. Assuming a localized image perturbation, Figure 3(b), we can compute slowness perturbations using the adjoint scattering operators, as defined in the preceding sections. Figure 7(a) shows the slowness perturbation for the zero-offset case and Figure 8(a) shows the similar slowness perturbation for the shot-record case. Here, too, we see slowness perturbations distributed in an area connecting the reflector to the surface, but in this case, there is no relative focus of the anomaly because the image perturbation is strictly localized on the reflector. For the shot-record case, the back-projection splits toward the source and receivers, corresponding to the upward continuation of the source and receiver wavefields. This case corresponds to the case of practical MVA where measurements of defocusing features are made on the image itself.

As we have done in the preceding experiment, we can also compute image perturbations using the forward scattering operators based on the back-projections created using the adjoint scattering operators. Figure 7(b) shows the image perturbation for the zero-offset case computed from the slowness perturbation in Figure 7(a) and Figure 8(b) shows the similar image perturbation for the shot-record case computed from the slowness perturbation in Figure 8(a). We can observe that the resulting image perturbations spread beyond the original location, indicating wider sensitivity of the wave-based MVA kernels to image perturbations than that of the corresponding ray-based MVA kernels.

Similar sensitivity can be observed for the more complex Sigsbee 2A model (Paffenholz et al., 2002), Figures 9(a)-9(b). Similarly to the preceding example, we simulate zero-offset data, Figure 11(a) and one shot corresponding to horizontal position...
Figures 12(a) and 13(a) correspond to the image perturbations for the slowness anomaly shown in Figure 10(a). We can observe image perturbations that spread in the vicinity of the reflector, similarly to the simpler example described earlier. The multi-pathing from the source to the reflector generates the multiple events characterizing the image perturbations. Figures 12(b) and 13(b) correspond to the slowness perturbations constructed by applying the zero-offset and shot-record adjoint scattering operators to the image perturbations from Figures 12(a) and 13(a). We see similar back-projection patterns to the ones observed in the preceding example, except that the propagation patterns are more complicated due to the presence of the salt body in the background model.

Figures 14(a) and 15(a) correspond to the slowness perturbations for the image anomaly shown in Figure 10(b). We can observe slowness perturbations that spread in the vicinity of the reflector, similarly to the simpler example described earlier. Finally, Figures 14(b) and 15(b) correspond to the image perturbations for the slowness perturbations constructed by the adjoint MVA operators shown in Figures 14(a) and 15(a) for the zero-offset and shot-record cases, respectively.

CONCLUSIONS

The wave-equation MVA operator discussed in this paper, can be implemented in various imaging frameworks, e.g. zero-offset (exploding reflector), survey-sinking or shot-record. In all cases, the forward and adjoint operators follow similar patterns involving combinations of scattering, imaging and extrapolation. The forward and adjoint operators share common elements and can be implemented in the mixed space-wavenumber domain, similarly to the implementation of the wavefield extrapolation operators.

The real challenges in using wave-based MVA are two-fold. First, the image perturbations need to be generated by techniques that do not compare image features that are too far from one-another, which is a property partially addressed by techniques based on differential semblance. Second, the cost of the wave-equation MVA operator is large, therefore a feasible implementation requires clever numeric implementation, e.g. by frequency decimation similarly to the approach taken in waveform inversion.

The examples shown in this paper illustrate the main characteristics of the various wave-equation MVA operators, i.e. stability during back-projection in background models with sharp velocity variation (e.g. salt), natural ability to characterize multi-pathing and wide area of sensitivity which is commensurate with the frequency band of the recorded data.
Figure 9: (a) Sigsbee 2A synthetic model and (b) a sub-salt horizontal reflector.
Figure 10: (a) Slowness perturbations used to demonstrate the WEMVA operators in Figures 12(a)-13(b) and (b) image perturbation used to demonstrate the WEMVA operators in Figures 14(a)-15(b).
Figure 11: (a) Simulated zero-offset data and (b) simulated shot-record data for the model depicted in Figures 9(a)-9(b) with a source located at coordinates $x = 14.6$ km and $z = 1.52$ km.
Figure 12: (a) Zero-offset image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from Figure 10(a) and (b) zero-offset slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from panel (a).
Figure 13: (a) Shot-record image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from Figure 10(a) and (b) shot-record slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from panel (a).

[geo2008NumericWEMVAoperators/saltWEMVA SFds2,SAFds2]
Figure 14: (a) Zero-offset slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from Figure 10(b) and (b) zero-offset image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from panel (a).
Figure 15: (a) Shot-record slowness perturbation obtained by the application of the adjoint scattering operator to the image perturbation from Figure 10(a) and (b) shot-record image perturbation obtained by the application of the forward scattering operator to the slowness perturbation from panel (a).
ACKNOWLEDGMENTS

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APPENDIX A

Mixed-domain operators

For the case of the phase-shift operation in media with lateral slowness variation, the mixed-domain solution involves forward and inverse Fourier transforms (denoted fFT and iFT in our algorithms) which can be implemented efficiently using standard Fast Fourier Transform algorithms. The numeric implementation is summarized in the following table:

<table>
<thead>
<tr>
<th></th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>fFT</td>
<td>$u(m) \overset{2D}{\rightarrow} u(k_m)$</td>
</tr>
<tr>
<td>$\omega - k$</td>
<td>$u(k_m) \ast = e^{\pm i k_z \Delta z}$</td>
</tr>
<tr>
<td>iFT</td>
<td>$u(m) \overset{2D}{\leftarrow} u(k_m)$</td>
</tr>
<tr>
<td>$\omega - x$</td>
<td>$u(m) \ast = e^{\pm i k_x \Delta z}$</td>
</tr>
</tbody>
</table>

In this chart, $k_z^k$ denotes the $\omega - k$ component of the depth wavenumber and $k_z^x$ denotes the $\omega - x$ component of the depth wavenumber. An example of mixed-domain implementation is the Split-Step Fourier (SSF) method, where $k_z^k$ represents the SSR equation computed with a constant reference slowness $\tilde{s}$, and $k_z^x = \omega (s - \tilde{s})$ represents a space-domain correction (Stoffa et al., 1990).

Based on the equation [27], the derivative of the depth wavenumber relative to slowness is

$$\frac{d k_z}{d s} \bigg|_{s_0} = \frac{\omega}{\sqrt{1 - \left[ \frac{|k_{m0}|}{\omega s_0(m)} \right]^2}}.$$

(A-1)

The numeric implementation of the pseudo-differential equation [A-1] is as complicated in media with lateral slowness variation as its phase-shift counterpart (equation 2). However, we can construct efficient and robust numeric implementations using similar approximations as the ones employed for the phase-shift relation, e.g. mixed-domain numeric implementation.

The linearized scattering operator can also be implemented in a mixed-domain by expanding the square-root from relation [A-1] using a Taylor series expansion

$$\frac{d k_z}{d s} \bigg|_{s_0} \approx \omega \left( 1 + \sum_{j=1}^{N} c_j \left[ \frac{|k_m|}{\omega s_0(m)} \right]^{2j} \right),$$

(A-2)

where $c_j$ are binomial coefficients of the Taylor series.
Therefore, the wavefield perturbation at depth $z$ caused by a slowness perturbation at depth $z$ under the influence of the background wavefield at the same depth $z$ (forward scattering operator $8$) can be written as

$$
\Delta u (m) \approx \pm i \omega \Delta z \left( 1 + \sum_{j=1}^{N} c_j \left[ \frac{|k_m|}{\omega s_0 (m)} \right]^{2j} \right) u (m) \Delta s (m). \quad (A-3)
$$

Similarly, the slowness perturbation at depth $z$ caused by a wavefield perturbation at depth $z$ under the influence of the background wavefield at the same depth $z$ (adjoint scattering operator $11$) can be written as

$$
\Delta s (m) \approx \mp i \omega \Delta z \left( 1 + \sum_{j=1}^{N} c_j \left[ \frac{|k_m|}{\omega s_0 (m)} \right]^{2j} \right) \overline{u (m) \Delta u (m)}. \quad (A-4)
$$

The mixed-domain implementation of the forward and adjoint scattering operators $A-3$ and $A-4$ is summarized on the following tables:

---

**MIXED-DOMAIN IMPLEMENTATION OF THE FORWARD SCATTERING OPERATOR $F_{ZOM}^{\pm}$**

\[
\begin{align*}
\Delta w (m) &= \Delta s (m) u (m) \\
\Delta p (m) &= \Delta w (m) \\
\text{fFT} \\
\Delta p (m) &\xrightarrow{2D} \Delta p (k_m) \\
j = 1 \ldots N \\
\omega - k \quad \Delta q (k_m) &= \Delta p (k_m) \\
\omega - x \quad \Delta q (m) &= \sum_{j=1}^{N} c_j \left[ \frac{|k_m|}{\omega s_0 (m)} \right]^{2j} \\
\omega - x \quad \Delta q (m) &= \Delta q (m) \\
\omega - x \quad \Delta w (m) &= \pm i \omega \Delta z
\end{align*}
\]

---

**MIXED-Domain IMPLEMENTATION OF THE ADJOINT SCATTERING OPERATOR $A_{ZOM}^{\mp}$**
\[
\begin{align*}
\Delta s(m) &= \Delta w(m) u(m) \\
\Delta p(m) &= \Delta s(m) \\
\text{fFT} & \quad \Delta p(m) \xrightarrow{2D} \Delta p(k_m) \\
& \quad j = 1 \ldots N \{ \\
& \quad \quad \Delta q(k_m) = \Delta p(k_m) \\
\omega - k & \quad \Delta q(k_m) * = |k_m|^{2j} \\
\text{iFT} & \quad \Delta q(m) \xleftarrow{2D} \Delta q(k_m) \\
\omega - x & \quad \Delta q(m) * = \frac{c_j}{|\omega s_0(m)|^j} \\
& \quad \Delta s(m) + = \Delta q(m) \\
\} \\
\Delta s(m) * &= \mp i\omega \Delta z
\end{align*}
\]
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Interferometric imaging condition for 
wave-equation migration

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and 
Oleg Poliannikov (Massachusetts Institute of Technology)

ABSTRACT

The fidelity of depth seismic imaging depends on the accuracy of the velocity models used for wavefield reconstruction. Models can be decomposed in two components corresponding to large scale and small scale variations. In practice, the large scale velocity model component can be estimated with high accuracy using repeated migration/tomography cycles, but the small scale component cannot. When the Earth has significant small-scale velocity components, wavefield reconstruction does not completely describe the recorded data and migrated images are perturbed by artifacts.

There are two possible ways to address this problem: improve wavefield reconstruction by estimating more accurate velocity models and image using conventional techniques (e.g. wavefield cross-correlation), or reconstruct wavefields with conventional methods using the known background velocity model, but improve the imaging condition to alleviate the artifacts caused by the imprecise reconstruction, which is what we suggest in this paper.

We describe the unknown component of the velocity model as a random function with local spatial correlations. Imaging data perturbed by such random variations is characterized by statistical instability, i.e. various wavefield components image at wrong locations that depend on the actual realization of the random model. Statistical stability can be achieved by pre-processing the reconstructed wavefields prior to the imaging condition. We employ Wigner distribution functions to attenuate the random noise present in the reconstructed wavefields, parametrized as a function of image coordinates. Wavefield filtering using Wigner distribution functions and conventional imaging can be lumped-together into a new form of imaging condition which we call an “interferometric imaging condition” due to its similarity to concepts from recent work on interferometry. The interferometric imaging condition can be formulated both for zero-offset and for multi-offset data, leading to robust and efficient imaging procedures that are effective in attenuating imaging artifacts due to unknown velocity models.

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INTRODUCTION

Seismic imaging in complex media requires accurate knowledge of the medium velocity. Assuming single scattering (Born approximation), imaging requires propagation of the recorded wavefields from the acquisition surface, followed by the application of an imaging condition highlighting locations where backscattering occurs, i.e. where reflectors are present. Typically, this is achieved with simple image processing techniques, e.g. cross-correlation of wavefields reconstructed from sources and receivers.

The main requirement for good-quality imaging is accurate knowledge of the velocity model. Errors in the model used for imaging lead to inaccurate reconstruction of the seismic wavefields and to distortions of the migrated images. In any realistic seismic field experiment the velocity model is never known exactly. Migration velocity analysis estimates large scale approximations of the model, but some fine scale variations always remain elusive. For example, when geology includes complicated stratigraphic structures or complex salt/carbonate bodies, the rapid velocity variations on the scale of the seismic wavelength and smaller cannot be estimated correctly by kinematic methods. Therefore, even if the broad kinematics of the seismic wavefields are reconstructed correctly, the extrapolated wavefields also contain phase and amplitude distortions that lead to image artifacts obstructing the image of the geologic structure under consideration. While it is certainly true that even the recovery of a long-wave background may prove to be a challenge in some circumstances, we do not attempt to address that issue in this paper. Instead, we concentrate solely on the problem of dealing with the effect of a small scale random variations not estimated by conventional methods.

There are two ways in which we can approach this problem. The first option is to improve our velocity analysis methods to estimate the small-scale variations in the model. Such techniques take advantage of all information contained in seismic wavefields and are not limited to kinematic information of selected events picked from the data. Examples of techniques in this category are waveform inversion (Taratola, 1987; Pratt and Worthington, 1990; Pratt, 1990; Sirgue and Pratt, 2004), wave-equation tomography (Woodward, 1992) or wave-equation migration velocity analysis (Sava and Biondi, 2004a,b; Shen et al., 2005). A more accurate velocity model allows for more accurate wavefield reconstruction. Then, wavefields can be used for imaging using conventional procedures, e.g. cross-correlation. The second option is to concentrate on the imaging condition, rather than concentrate on “perfect” wavefield reconstruction. Assuming that the large-scale component of the velocity models is known (e.g. by iterative migration/tomography cycles), we can design imaging conditions that are not sensitive to small inaccuracies of the reconstructed wavefields. Imaging artifacts can be reduced at the imaging condition step, despite the fact that the wavefields incorporate small kinematic errors due to velocity fluctuations.

Of course, the two options are complementary to each other, and both can contribute to imaging accuracy. In this paper, we concentrate on the second approach. For purposes of theoretical analysis, it is convenient to model the small-scale velocity
fluctuations as random but spatially correlated variations superimposed on a known velocity. We assume that we know the background model, but that we do not know the random fluctuations. The goal is to design an imaging condition that alleviates artifacts caused by those random fluctuations. Conventional imaging consists of cross-correlations of extrapolated source and receiver wavefields at image locations. Since wavefield extrapolation is performed using an approximation of the true model, the wavefields contain random time delays, or equivalently random phases, which lead to imaging artifacts.

One way of mitigating the effects of the random model on the quality of the resulting image is to use techniques based on acoustic time reversal (Fink, 1999). Under certain assumptions, a signal sent through a random medium, recorded by a receiver array, time reversed and sent back through the same medium, refocuses at the source location in a statistically stable fashion. Statistical stability means that the refocusing properties (i.e. image quality) are independent of the actual realization of the random medium (Papanicolaou et al., 2004; Fouque et al., 2005).

We investigate an alternative way of increasing imaging statistical stability. Instead of imaging the reconstructed wavefields directly, we first apply a transformation based on Wigner distribution functions (Wigner, 1932) to the reconstructed wavefields. We consider a special case of the Wigner distribution function (WDF) which has the property that it attenuates random fluctuations from the wavefields after extrapolation with conventional techniques. The idea for this method is borrowed from image processing where WDFs are used for filtering of random noise. Here, we apply WDFs to the reconstructed wavefields, prior to the imaging condition. This is in contrast to data filtering prior to wavefield reconstruction or to image filtering after the application of an imaging condition.

Our procedure closely resembles conventional imaging procedures where wavefields are extrapolated in the image volume and then cross-correlated in time at every image location. Our method uses WDFs defined in three-dimensional windows around image locations which makes it both robust and efficient. From an implementation and computational cost point of view, our technique is similar to conventional imaging, but its statistical properties are improved. Although conceptually separate, we can lump-together the WDF transformation and conventional imaging into a new form of imaging condition which resembles interferometric techniques (Papanicolaou et al., 2004; Fouque et al., 2005). Therefore, we use the name interferometric imaging condition for our technique to contrast it with the conventional imaging condition.

A related method discussed in the literature is known under the name of coherent interferometric imaging (Borcea et al., 2006a,b,c). This method uses similar local cross-correlations and averaging, but unlike our method, it parametrizes reconstructed wavefields as a function of receiver coordinates. Thus, the coherent interferometric imaging functional requires separate wavefield reconstruction from every receiver position, which makes this technique prohibitively expensive and probably unusable in practice on large-scale seismic imaging projects. In contrast, the imaging technique advocated in this paper achieves similar statistical stability properties as
coherent interferometric imaging, but at an affordable computational cost since we apply wavefield reconstruction only once for all receiver locations corresponding to a given seismic experiment, typically a “shot”.

**IMAGING CONDITIONS**

**Conventional imaging condition**

Let $D(\mathbf{x}, t)$ be the data recorded at time $t$ at receivers located at coordinates $\mathbf{x}$ for a seismic experiment with buried sources (Figure 1(a), also known as an exploding reflector seismic experiment [Loewenthal et al., 1976]. A conventional imaging procedure for this type of data consists of two steps (Claerbout, 1985): wavefield reconstruction at image coordinates $\mathbf{y}$ from data recorded at receiver coordinates $\mathbf{x}$, followed by an imaging condition taking the reconstructed wavefield at time $t = 0$ as the seismic image.

![Figure 1: Zero-offset seismic experiment sketch (a) and multi-offset seismic experiment sketch (b). Coordinates $\mathbf{x} = \{x, y\}$ characterize receiver positions on the surface and coordinates $\mathbf{y} = \{x, y, z\}$ characterize reflector positions in the subsurface.](geo2008InterferometricImagingCondition/XFig paraarray,aarray)

Mathematically, we can represent the wavefield $V$ reconstructed at coordinates $\mathbf{x}$ from data $D$ recorded at coordinates $\mathbf{x}$ as a temporal convolution of the recorded trace with the Green’s function $G$ connecting the two points:

$$V(\mathbf{x}, \mathbf{y}, t) = D(\mathbf{x}, t) *_t G(\mathbf{x}, \mathbf{y}, t) .$$  

The total wavefield $U$ reconstructed at $\mathbf{y}$ from all receivers is the superposition of the
wavefields $V$ reconstructed from individual traces

$$U(y,t) = \int dx \, V(x,y,t),$$

(2)

where the integral over $x$ spans the entire receiver space. As stated, the imaging condition extracts the image $R(y)$ from the wavefield $U(y,t)$ at time $t = 0$, i.e.

$$R(y) = U(y,t=0).$$

(3)

The Green’s function used in the procedure described by equation 1 can be implemented in different ways. For our purposes, the actual method used for computing Green’s functions is not relevant. Any procedure can be used, although different procedures will be appropriate in different situations, with different cost of implementation. We assume that a satisfactory procedure exists and is appropriate for the respective velocity models used for the simulations. In our examples, we compute Green’s functions with time-domain finite-difference solutions to the acoustic wave-equation, similar to reverse-time migration [Baysal et al. 1983].

Consider the velocity model depicted in Figure 2(a) and the imaging target depicted in Figure 2(b). We assume that the model with random fluctuations (Figure 3(b)) represents the real subsurface velocity and use this model to simulate data. We consider the background model (Figure 3(a)) to represent the migration velocity and use this model to migrate the data simulated in the random model. We consider one source located in the subsurface at coordinates $z = 8$ km and $x = 13.5$ km, and receivers located close to the top of the model at discrete horizontal positions and depth $z = 0.0762$ km. Figure 3(c)-3(d) show snapshots of the simulated wavefields at a later time. The panels on the left correspond to modeling in the background model, while the panels on the right correspond to modeling in the random model.

Figure 3(f) shows the data recorded on the surface. The direct wavefield arrival from the seismic source is easily identified in the data, although the wavefronts are
Figure 3: Seismic snapshots of acoustic wavefields simulated in the background velocity model (a)-(c), and in the random velocity model (b)-(d). Data recorded at the surface from simulation in the background velocity model (e) and from the simulation in the random velocity model (f).
distorted by the random perturbations in the medium. For comparison, Figure 3(e) shows data simulated in the background model, which do not show random fluctuations.

Conventional imaging using the procedure described above implicitly states that data generated in the random model, Figure 3(b), are processed as if they were generated in the background model, Figure 3(a). Thus, the random phase variations in the data are not properly compensated during the imaging procedure causing artifacts in the image. Figure 6(b) shows the image obtained by migrating data from Figure 3(f) using the model from Figure 3(a). For comparison, Figure 6(a) shows the image obtained by migrating the data from Figure 3(e) using the same model from Figure 3(a).

Ignoring aperture effects, the artifacts observed in the images are caused only by the fact that the velocity models used for modeling and migration are not the same. Small artifacts caused by truncation of the data on the acquisition surface can also be observed, but those artifacts are well-known (i.e. truncation butterflies) and are not the subject of our analysis. In this example, the wavefield reconstruction procedure is the same for both modeling and migration (i.e. time-domain finite-difference solution to the acoustic wave-equation), thus it is not causing artifacts in the image. We can conclude that the migration artifacts are simply due to the phase errors between the Green’s functions used for modeling (with the random velocity) and the Green’s functions used for migration (with the background velocity). The main challenge for imaging in media with random variations is to design procedures that attenuate the random phase delays introduced in the recorded data by the unknown variations of the medium without damaging the real reflections present in the data.

The random phase fluctuations observed in recorded data (Figure 3(f)) are preserved during wavefield reconstruction using the background velocity model. We can observe the randomness in the extrapolated wavefields in two ways, by reconstructing wavefields using individual data traces separately, or by reconstructing wavefields using all data traces at once.

The first option is to reconstruct the seismic wavefield at all image locations $y$ from individual receiver positions on the surface. Of course, this is not conventionally done in reverse-time imaging, but we describe this concept just for illustration purposes. Figure 4(a) shows the wavefield reconstructed separately from individual data traces depicted in Figure 3(f) using the background model depicted in Figure 3(b). In Figure 4(a), the horizontal axis corresponds to receiver positions on the surface, i.e. coordinates $x$, and the vertical axis represents time. According to the notations used in this paper, Figure 4(a) shows the wavefield $V(x, y, t)$ reconstructed to a particular image coordinate $y$ from separate traces located on the surface at coordinates $x$. A similar plot can be constructed for all other image locations. Ideally, the reconstructed wavefield should line-up at time $t = 0$, but this is not what we observe in this figure, indicating that the input data contain random phase delays that are not compensated during wavefield reconstruction using the background velocity.
The second option is to reconstruct the seismic wavefield at all image locations $y$ from all receiver positions on the surface at once. This is a conventional procedure for reverse-time imaging. Figure 5(a) shows the wavefield reconstructed from all data traces depicted in Figure 3(f) using the background model depicted in Figure 3(b). In Figure 5(a), the vertical and horizontal axes correspond to depth and horizontal positions around the source, i.e. coordinates $y$, and the third cube axis represents time. According to the notations used in this paper, Figure 5(a) corresponds to wavefield $U(y,t)$ reconstructed from data at all receiver coordinates $x$ to image coordinates $y$. The reconstructed wavefield does not focus completely at the image coordinate and time $t = 0$ indicating that the input data contains random phase delays that are not compensated during wavefield reconstruction using the background velocity.

Wigner distribution functions

One possible way to address the problem of random fluctuations in reconstructed wavefields is to use Wigner distribution functions (Wigner, 1932) to pre-process the wavefields prior to the application of the imaging condition. Appendix C provides a brief introduction for readers unfamiliar with Wigner distribution functions. More details about this topic are presented by Cohen (1995).

Wigner distribution functions (WDF) are bi-linear representations of multi-dimensional signals defined in phase space, i.e. they depend simultaneously on position-wavenumber $(y - k)$ and time-frequency $(t - \omega)$. Wigner (1932) developed these concepts in the context of quantum physics as probability functions for the simultaneous description of coordinates and momenta of a given wave function. WDFs were introduced to signal processing by Ville (1948) and have since found many applications in signal and image processing, speech recognition, optics, etc.

A variation of WDFs, called pseudo Wigner distribution functions are constructed using small windows localized in space and/or time (Appendix C). Pseudo WDFs are simple transformations with efficient application to multi-dimensional signals. In this paper, we apply the pseudo WDF transformation to multi-dimensional seismic wavefields obtained by reconstruction from recorded seismic data. We use pseudo WDFs for decomposition and filtering of extrapolated space-time signals as a function of their local wavenumber-frequency. In particular, pseudo WDFs can filter reconstructed wavefields to retain their coherent components by removing high-frequency noise associated with random fluctuations in the wavefields due to random fluctuations in the model.

The idea for our method is simple: instead of imaging the reconstructed wavefields directly, we first filter them using pseudo WDFs to attenuate the random phase noise, and then proceed to imaging using a conventional or an extended imaging conditions. Wavefield filtering occurs during the application of the zero-frequency end-member of the pseudo WDF transformation, which reduces the random character of the field. For the rest of the paper, we use the abbreviation WDF to denote this special case.
of pseudo Wigner distribution functions, and not its general form.

As we described earlier, we can distinguish two options. The first option is to use wavefield parametrization as a function of data coordinates \( x \). In this case, we can write the pseudo WDF of the reconstructed wavefield \( V(x, y, t) \) as

\[
V_x(x, y, t) = \int_{|t_h| \leq T} \int_{|x_h| \leq X} \left[ V \left( x - \frac{x_h}{2}, y, t - \frac{t_h}{2} \right) V \left( x + \frac{x_h}{2}, y, t + \frac{t_h}{2} \right) \right],
\]

where \( x_h \) and \( t_h \) are variables spanning space and time intervals of total extent \( X \) and \( T \), respectively. For 3D surface acquisition geometry, the 2D variable \( x_h \) is defined on the acquisition surface. The second option is to use wavefield parametrization as a function of image coordinates \( y \). In this case, we can write the pseudo WDF of the reconstructed wavefield \( U(y, t) \) as

\[
W_y(y, t) = \int_{|t_h| \leq T} \int_{|y_h| \leq Y} \left[ U \left( y - \frac{y_h}{2}, t - \frac{t_h}{2} \right) U \left( y + \frac{y_h}{2}, t + \frac{t_h}{2} \right) \right],
\]

where \( y_h \) and \( t_h \) are variables spanning space and time intervals of total extent \( Y \) and \( T \), respectively. For 3D surface acquisition geometry, the 3D variable \( y_h \) is defined around image positions.

For the examples used in this section, we employ 41 grid points for the interval \( X \) centered around a particular receiver position, \( 5 \times 5 \) grid points for the interval \( Y \) centered around a particular image point, and 21 grid points for the interval \( T \) centered around a particular time. These parameters are not necessarily optimal for the transformation, since they characterize the local WDF windows and depend on the specific implementation of the pseudo WDF transformation. The main criterion used for selecting the size of the space-time window for the pseudo WDF transformation is that of avoiding cross-talk between nearby events, e.g. reflections. Finding the optimal size of this window is an important consideration for our method, although its complete treatment falls outside the scope of the current paper and we leave it for future research. Preliminary results on optimal window selection are discussed by Borcea et al. (2006a).

Figure 4(b) depicts the results of applying the pseudo WDF transformation to the reconstructed wavefield in Figure 4(a). For the case of modeling in the random model and reconstruction in the background model, the pseudo WDF attenuates the random character of the wavefield significantly, Figure 4(b). The random character of the reconstructed wavefield is reduced and the main events cluster more closely around time \( t = 0 \). Similarly, Figure 5(b) depicts the results of applying the pseudo WDF transformation to the reconstructed wavefields in Figure 5(a). For the case of modeling in the random model and reconstruction in the background model, the pseudo WDF also attenuates the random character of the wavefield significantly, Figure 5(b). The random character of the reconstructed wavefields is also reduced and the main events focus at the correct image location at time \( t = 0 \).
Figure 4: Reconstructed seismic wavefield as a function of data coordinates (a) and its pseudo Wigner distribution function (b) computed as a function of data coordinates $x$ and time $t$. The wavefield is reconstructed using the background model from data simulated in the random model.

Figure 5: Reconstructed seismic wavefield as a function of image coordinates (a) and its pseudo Wigner distribution function (b) computed as a function of image coordinates $y$ and time $t$. The wavefield is reconstructed using the background model from data modeled in the random model.
Zero-offset interferometric imaging condition

After filtering the reconstructed wavefields with pseudo WDFs, we can perform imaging with normal procedures. For the case of wavefields parametrized as a function of data coordinates, we obtain the total wavefield at image coordinates by summing over receiver coordinates $\mathbf{x}$

$$W_x(y,t) = \int dx \ V_x(x,y,t), \quad (6)$$

followed by a conventional imaging condition extracting time $t = 0$ from the pseudo WDF of the reconstructed wavefields:

$$R_x(y) = W_x(y,t=0). \quad (7)$$

The image obtained with this imaging procedure is shown in Figure 6(c). As expected, the artifacts caused by the unknown random fluctuations in the model are reduced, leaving a cleaner image of the source.

Similarly, for the case of wavefields parametrized as a function of image coordinates, we obtain the image by application of the conventional imaging condition extracting time $t = 0$ from the pseudo WDF of the reconstructed wavefield:

$$R_y(y) = W_y(y,t=0). \quad (8)$$

The image obtained with this imaging procedure is shown in Figure 6(d). As in the preceding case, the artifacts caused by the unknown random fluctuations in the model are reduced, producing a cleaner image of the source, comparable with the one in Figure 6(c).

Figure 6: Images produced by the conventional imaging condition using the data simulated in the background model (a) and using the data simulated in the random model (b). Images produced from data simulated in the random model using the interferometric imaging condition with parametrization as a function of data coordinates (c) and as a function of image coordinates (d).
Multi-offset interferometric imaging condition

The imaging procedure in equations 5-8 can be generalized for imaging prestack (multi-offset) data (Figure 1(b)). The conventional imaging procedure for this type of data consists of two steps (Claerbout, 1985): wavefield simulation from the source location to the image coordinates \( y \) and wavefield reconstruction at image coordinates \( y \) from data recorded at receiver coordinates \( x \), followed by an imaging condition evaluating the match between the simulated and reconstructed wavefields.

Let \( U_S(y, t) \) be the source wavefield constructed from the location of the seismic source and \( U_R(y, t) \) the receiver wavefield reconstructed from the receiver locations. A conventional imaging procedure produces a seismic image as the zero-lag of the time cross-correlation between the source and receiver wavefields. Mathematically, we can represent this operation as

\[
R(y) = \int dt \, U_S(y, t) U_R(y, t),
\]

where \( R(y) \) represents the seismic image for a particular seismic experiment at coordinates \( y \). When multiple seismic experiments are processed, a complete image is obtained by summation of the images constructed for individual experiments. The actual reconstruction methods used to produce the wavefields \( U_S(y, t) \) and \( U_R(y, t) \) are irrelevant for the present discussion. As in the zero-offset/exploding reflector case, we use time-domain finite-difference solutions to the acoustic wave-equation, but any other reconstruction technique can be applied without changing the imaging approach.

When imaging in random media, the data recorded at the surface incorporates phase delays caused by the velocity variations encountered while waves propagate in the subsurface. In a typical seismic experiment, random phase delays accumulate both on the way from the source to the reflectors, as well as on the way from the reflectors to the receivers. Therefore, the receiver wavefield reconstructed using the background velocity model is characterized by random fluctuations, similar to the ones seen for wavefields reconstructed in the zero-offset situation. In contrast, the source wavefield is simulated in the background medium from a known source position and, therefore, it is not affected by random fluctuations. However, the zero-lag of the cross-correlations between the source wavefields (without random fluctuations) and the receiver wavefield (with random fluctuations), still generates image artifacts similar to the ones encountered in the zero-offset case.

Statistically stable imaging using pseudo WDFs can be obtained in this case, too. What we need to do is attenuate the phase errors in the reconstructed receiver wavefield and then apply a conventional imaging condition. Therefore, a multi-offset interferometric imaging condition can be formulated as

\[
R(y) = \int dt \, U_S(y, t) W_R(y, t),
\]
where $W_R(y, t)$ represents the pseudo WDF of the receiver wavefield $U_R(y, t)$ which can be constructed, in principle, either with parametrization relative to data coordinates, according to equations 4-6 or relative to image coordinates, according to equation 5. Of course, our choice is to use image-space parametrization for computational efficiency reasons.

**Discussion**

The strategies described in the preceding section have notable similarities and differences. The imaging procedures 4-6-7 and 5-8 are similar in that they employ wavefields reconstructed from the surface data in similar ways. Neither method uses the surface recorded data directly, but they use wavefields reconstructed from those data as boundary conditions to numerical solutions of the acoustic wave-equation. The actual wavefield reconstruction procedure is identical in both cases.

The techniques are different because imaging with equations 4-6-7 employs independent wavefield reconstruction from receiver locations $x$ to image locations $y$. In practice, this requires separately solving the acoustic wave-equation, e.g. by time-domain finite-differences, from all receiver locations on the surface. Such computational effort is often prohibitive in practice. In contrast, imaging with equations 5-8 is similar to conventional imaging because it requires only one wavefield reconstruction using all recorded data at once, i.e. only one solution to the acoustic wave-equation, similar to conventional shot-record migration.

The techniques 4-6-7 and 5-8 are similar in that they both employ noise suppression using pseudo Wigner distribution functions. However, the methods are parametrized differently, the former relative to data coordinates with 2D local space averaging and the later relative to image coordinates with 3D local space averaging.

The imaging functionals presented in this paper are described as functions of space coordinates, $x$ or $y$, and time, $t$. As suggested in Appendix C, pseudo WDFs can be implemented either in time or frequency, so potentially the imaging conditions discussed in this paper can also be implemented in the frequency-domain. However, we restrict our attention in this paper to the time-domain implementation and leave the frequency-domain implementation subject to future study.

Equations 4-6-7 can be collected into the zero-offset imaging functional

$$R_{CINT}(y) = \delta(t) \int dx \int dt_h \int dx_h \left[ V\left(x - \frac{x_h}{2}, y, t - \frac{t_h}{2}\right) V\left(x + \frac{x_h}{2}, y, t + \frac{t_h}{2}\right) \right],$$  

where the temporal $\delta$ function implements the zero time imaging condition. A similar form can be written for the multi-offset case. Equation 11 corresponds to the time-domain version of the coherent interferometric functional proposed by Borcea et al. (2006a,b,c). Consistent with the preceding discussion, the cost required to implement
this imaging functional is often prohibitive for practical application to seismic imaging problems.

**STATISTICAL STABILITY**

The interferometric imaging condition described in the preceding section is used to reduce imaging artifacts by attenuating the incoherent energy corresponding to velocity errors, as illustrated in Figures 6(b) and 6(d). The random model used for this example corresponds to the weak fluctuation regime, as explained in Appendix A (characteristic wavelength of similar scale with the random fluctuations in the medium and fluctuations with small magnitude).

By statistical instability we mean that images obtained for different realizations of random models with the identical statistics are different. Figures 7(a)-7(c) illustrate data modeled for different realizations of the random model in Figure 3(b). The general kinematics of the data are the same, but subtle differences exist between the various datasets due to the random model variations. Migration using a conventional imaging condition leads to the images in Figures 7(d)-7(f) which also show variations from one realization to another. In contrast, Figures 7(g)-7(i) show images obtained by the interferometric imaging condition in equations 5-8 which are more similar to one-another since many of the artifacts have been attenuated.

In typical seismic imaging problems, we cannot ensure that random velocity fluctuations are small (e.g. \( \sigma \leq 5\% \)). It is desirable that imaging remains statistically stable even in cases when velocity varies with larger magnitude. We investigate the statistical properties of the imaging functional in equations 5-8 using numerical experiments similar to the one used earlier. We describe the random noise present in the velocity models using the following parameters explained in Appendix A: seismic spatial wavelength \( \lambda = 76.2 \) m, wavelet central frequency \( \omega = 20 \) Hz, random fluctuations parameters: \( r_a = 0.0762, r_c = 0.0762, \alpha = 2 \), and random noise magnitude \( \sigma \) between 15% and 45%. This numerical experiment simulates a situation that mixes the theoretical regimes explained in Appendix B: random model fluctuations of comparable scale with the seismic wavelength lead to destruction of the wavefronts, as suggested by the “weak fluctuations” regime; large magnitude of the random noise leads to diffusion of the wavefronts, as suggested by the “diffusion approximation” regime. This combination of parameters could be regarded as a worst-case-scenario from a theoretical standpoint.

Figures 8(a)-8(c) show data simulated in models similar to the one depicted in Figure 3(b), but where the random noise component is described by \( \sigma = 15, 30, 45\% \), respectively. As expected, the wavefronts recorded at the surface are increasingly distorted to the point where some of the later arrival are not even visible in the data.

Migration using a conventional imaging condition leads to the images in Figures 8(d)-8(f). As expected, the images show stronger artifacts due to the larger defocusing caused by the unknown random fluctuations in the model. However,
Figure 7: Illustration of statistical stability for the interferometric imaging condition in presence of random model variations. Data modeled using velocity with random variations of magnitude $\sigma = 30\%$, for different realizations of the noise model $n$. Images obtained by conventional imaging (d)-(f) and images obtained by interferometric imaging (g)-(i).
migration using the interferometric imaging condition leads to the images in Figures 8(g)-8(i). Artifacts are significantly reduced and the images are much better focused.

**MULTI-OFFSET IMAGING EXAMPLES**

There are many potential applications for this interferometric imaging functional. One application we illustrate in this paper is imaging of complex stratigraphy through a medium characterized by unknown random variations. In this situation, accurate imaging using conventional methods requires velocity models that incorporate the small scale (random, as we view them) velocity variations. However, practical migration velocity analysis does not produce models of this level of accuracy, but approximates them with smooth, large-scale fluctuations one order of magnitude larger than that of the typical seismic wavelength. Here, we study the impact of the unknown (random) component of the velocity model on the images and whether interferometric imaging increases the statistical stability of the image.

For all our examples, we extrapolate wavefields using time-domain finite-differences both for modeling and for migration. Thus, we simulate a reverse-time imaging procedure, although the theoretical results derived in this paper apply equally well to other wavefield reconstruction techniques, e.g. downward continuation, Kirchhoff integral methods, etc. The parameters used in our examples, explained in Appendix A, are: seismic spatial wavelength $\lambda = 76.2$ m, wavelet central frequency $\omega = 20$ Hz, random fluctuations parameters: $r_a = 76.2$ m, $r_c = 76.2$ m, $\alpha = 2$, and random noise magnitude $\sigma = 20\%$.

Consider the model depicted in Figures 10(a)-10(d). As in the preceding example, the left panels depict the known smooth velocity $v_0$, and the right panels depict the model with random variations. The imaging target is represented by the oblique lines, Figure 9(b), located around $z = 8$ km, which simulates a cross-section of a stratigraphic model.

We model data with a random velocity model and image using the smooth model. Figures 10(a)-10(d) show wavefield snapshots in the two models for different propagation times, one before the source wavefields interact with the target reflectors and one after this interaction. The propagating waves are affected the the random fluctuations in the model both before and after their interaction with the reflectors. Figures 10(e) and 10(f) show the corresponding recorded data on the acquisition surface located at $z = \lambda$, where $\lambda$ represents the wavelength of the source pulse.

Migration with a conventional imaging condition of the data simulated in the background model using the same velocity produces the image in Figure 11(a). The targets are well imaged, although the image also shows artifacts due to truncation of the data on the acquisition surface. In contrast, migration with the conventional imaging condition of the data simulated in the random model using the background velocity produces the image in Figure 11(b). This image is distorted by the random
Figure 8: Illustration of interferometric imaging condition robustness in presence of random model variations. Data modeled using velocity with random variations of magnitudes $\sigma = 15\%$ (a), $\sigma = 30\%$ (b), and $\sigma = 45\%$ (c). Images obtained by conventional imaging (d)-(f) and images obtained by interferometric imaging (g)-(i).
variations in the model that are not accounted for in the background migration velocity. The targets are harder to discern since they overlap with many truncation and defocusing artifacts caused by the inaccurate migration velocity.

Finally, Figure 11(c) shows the migrated image using the interferometric imaging condition applied to the wavefields reconstructed in the background model from the data simulated in the random model. Many of the artifacts caused by the inaccurate velocity model are suppressed and the imaging targets are more clearly visible and easier to interpret. Furthermore, the general patterns of amplitude variation along the imaged reflectors are similar between Figures 11(b) and 11(c).

We note that the reflectors are not as well imaged as the ones obtained when the velocity is perfectly known. This is because the interferometric imaging condition described in this paper does not correct kinematic errors due to inaccurate velocity. It only acts on the extrapolated wavefields to reduce wavefield incoherency and add statistical stability to the imaging process. Further extensions to the interferometric imaging condition can improve focusing and enhance the images by correcting wavefields prior to imaging. However, this topic falls outside the scope of this paper and we do not elaborate on it further.

Figure 9: Velocity model (a) and imaging target located around $z = 8$ km (b). The model consists of a smooth version of the Sigsbee2A velocity, in order to avoid backscattering during reverse-time migration. The shaded area is imaged in Figures 11(a) and 11(c).

CONCLUSIONS

We extend the conventional seismic imaging condition based on wavefield cross-correlations to achieve statistical stability for models with rapid, small-scale velocity variation. We assume that the random velocity variations on a scale comparable with the seismic wavelength are modeled by correlated Gaussian distributions. Our proposed interferometric imaging condition achieves statistical stability by applying conventional imaging to the Wigner distribution functions of the reconstructed seismic wavefields. The interferometric imaging condition is a natural extension of the cross-correlation imaging condition and adds minimally to the cost of migration. The main characteristic of the method is that it operates on extrapolated wavefields at
Figure 10: Seismic snapshots of acoustic wavefields simulated in the background velocity model (a)-(c), and in the random velocity model (b)-(d). Data recorded at the surface from the simulation in the background velocity model (e) and from the simulation in the random velocity model (f).
Figure 11: Image produced using the conventional imaging condition from data simulated in the background model (a) and from data simulated in the random model (b). Image produced using the interferometric imaging condition from data simulated in the random model (c).
image positions (thus the name interferometric imaging condition), in contrast with costlier alternative approaches using interferometry parametrized as a function of receiver coordinates.

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REFERENCES

Noise model

Consider a medium whose behavior is completely defined by the acoustic velocity, i.e. assume that the density \( \rho (x, y, z) = \rho_0 \) is constant and the velocity \( v(x, y, z) \) fluctuates around a homogenized value \( v_0 (x, y, z) \) according to the relation

\[
\frac{1}{v^2(x, y, z)} = 1 + \frac{\sigma m(x, y, z)}{v_0^2(x, y, z)},
\]  
(A-1)

where the parameter \( m \) characterizes the type of random fluctuations present in the velocity model, and \( \sigma \) denotes their strength.

Consider the covariance orientation vectors

\[
a &= (a_x, a_y, a_z)^\top \in \mathbb{R}^3 \\
b &= (b_x, b_y, b_z)^\top \in \mathbb{R}^3 \\
c &= (c_x, c_y, c_z)^\top \in \mathbb{R}^3
\]  
(A-2) \quad (A-3) \quad (A-4)

defining a coordinate system of arbitrary orientation in space. Let \( r_a, r_b, r_c > 0 \) be the covariance range parameters in the directions of \( a, b, c \), respectively.

We define a covariance function

\[
cov(x, y, z) = \exp \left[ -l^\alpha (x, y, z) \right],
\]  
(A-5)

where \( \alpha \in [0, 2] \) is a distribution shape parameter and

\[
l(x, y, z) = \sqrt{\left( \frac{a \cdot r}{r_a} \right)^2 + \left( \frac{b \cdot r}{r_b} \right)^2 + \left( \frac{c \cdot r}{r_c} \right)^2}
\]  
(A-6)

is the distance from a point at coordinates \( r = (x, y, z) \) to the origin in the coordinate system defined by \( \{r_a a, r_b b, r_c c\} \).
Given the IID Gaussian noise field $n(x,y,z)$, we obtain the random noise $m(x,y,z)$ according to the relation

\[
m(x,y,z) = \mathcal{F}^{-1}\left[\sqrt{\hat{\text{cov}}(k_x,k_y,k_z)} \hat{n}(k_x,k_y,k_z)\right], \tag{A-7}
\]

where $k_x, k_y, k_z$ are wavenumbers associated with the spatial coordinates $x, y, z$, respectively. Here,

\[
\hat{\text{cov}} = \mathcal{F}[\text{cov}] \tag{A-8}
\]
\[
\hat{n} = \mathcal{F}[n] \tag{A-9}
\]

are Fourier transforms of the covariance function $\text{cov}$ and the noise $n$, $\mathcal{F}[\cdot]$ denotes Fourier transform, and $\mathcal{F}^{-1}[\cdot]$ denotes inverse Fourier transform. The parameter $\alpha$ controls the visual pattern of the field, and $a, b, c, r_a, r_b, r_c$ control the size and orientation of a typical random inhomogeneity.

\section*{APPENDIX B}

\section*{Wave propagation and scale regimes}

Acoustic waves characterized by pressure $p(x,y,z,t)$ propagate according to the second order acoustic wave-equation for constant density

\[
\frac{\partial^2 p}{\partial t^2} = v^2 \nabla^2 p + F_\lambda(t), \tag{B-1}
\]

where $F_\lambda(t)$ is a wavelet of characteristic wavelength $\lambda$.

Given the parameters $l$ (size of inhomogeneities), $\lambda$ (wavelength size), $L$ (propagation distance) and $\sigma$ (noise strength), we can define several propagation regimes.

The weak fluctuations regime characterized by waves with wavelength of size comparable to that of typical inhomogeneities propagating over a medium with small fluctuations to a distance of many wavelengths. This regime is characterized by negligible back scattering, and the randomness impacts the propagating waves through forward multipathing. The relevant length parameters are related by

\[
l \sim \lambda \ll L, \tag{B-2}
\]

and the noise strength is assumed small

\[
\sigma \ll 1. \tag{B-3}
\]

The diffusion approximation regime characterized by waves with wavelength much larger than that of typical inhomogeneities propagate over a medium with strong fluctuations to a distance of many wavelengths. This regime is characterized by
traveling waves that are statistically stable but diffuse with time. Back propagation of such waves in a medium without random fluctuations results in loss of resolution. The relevant length parameters are related by

\[ l \ll \lambda \ll L , \]

and the noise strength is not assumed small

\[ \sigma \sim 1 . \]

APPENDIX C

Wigner distribution functions

Consider the complex signal \( u (t) \) which depends on time \( t \). By definition, its Wigner distribution function (WDF) is \( [\text{Wigner} 1932] \):

\[
W (t, \omega) = \frac{1}{2\pi} \int u^* \left( t - \frac{\tau_h}{2} \right) u \left( t + \frac{\tau_h}{2} \right) e^{-i \omega \tau_h} d\tau_h ,
\]

\[ \text{(C-1)} \]

where \( \omega \) denotes temporal frequency, \( \tau_h \) denotes the relative time shift of the considered signal relative to a reference time \( t \) and the sign \( * \) denotes complex conjugation of complex signal \( s \). The same WDF can be obtained in terms of the spectrum \( U (\omega) \) of the signal \( u (t) \):

\[
W (t, \omega) = \frac{1}{2\pi} \int U^* \left( \omega + \frac{\omega_h}{2} \right) U \left( \omega - \frac{\omega_h}{2} \right) e^{-i \omega \omega_h} d\omega_h ,
\]

\[ \text{(C-2)} \]

where \( \omega_h \) denotes the relative frequency shift of the considered spectrum relative to a reference frequency \( \omega \). The time integral in equation \( \text{(C-1)} \) or the frequency integral in equation \( \text{(C-2)} \) spans the entire domain of time and frequency, respectively. When the interval is limited to a region around the reference value, the transformation is known as pseudo Wigner distribution function.

A special subset of the transformation equation \( \text{(C-1)} \) corresponds to zero temporal frequency. For input signal \( u (t) \), we obtain the output Wigner distribution function \( W (t) \) as

\[
W (t) = \frac{1}{2\pi} \int u^* \left( t - \frac{\tau_h}{2} \right) u \left( t + \frac{\tau_h}{2} \right) d\tau_h .
\]

\[ \text{(C-3)} \]

The WDF transformation can be generalized to multi-dimensional signals of space and time. For example, for 2D real signals function of space, \( u (x, y) \), the zero-wavenumber pseudo WDF can be formulated as

\[
W (x, y) = \frac{1}{4\pi^2} \int_{|x_h| \leq X} \int_{|y_h| \leq Y} u \left( x - \frac{x_h}{2}, y - \frac{y_h}{2} \right) u \left( x + \frac{x_h}{2}, y + \frac{y_h}{2} \right) dx_h dy_h ,
\]

\[ \text{(C-4)} \]
where $x_h$ and $y_h$ denote relative shift of the signal $s$ relative to positions $x$ and $y$, respectively. In this particular form, the pseudo WDF transformation has the property that it filters the input of random fluctuations preserving in the output image the spatially coherent components in a noise-free background.

For illustration, consider the model depicted in Figure 1(a). This model consists of a smoothly-varying background with 25% random fluctuations. The acoustic seismic wavefield corresponding to a source located in the middle of the model is depicted in Figure 1(b). This wavefield snapshot can be considered as the random “image”. The application of the 2D pseudo WDF transformation to images shown in Figure 1(b) produces the image shown in Figure 1(c). We can make three observations on this image: first, the random noise is strongly attenuated; second, the output wavelet is different from the input wavelet, as a result of the bi-linear nature of the pseudo WDF transformations; third, the transformation is isotropic, i.e. it operates identically in all directions. The pseudo WDF applied to this image uses $11 \times 11$ grid points in the vertical and horizontal directions. As indicated in the body of the paper, we do not discuss here the optimal selection of the WDF window. Further details of Wigner distribution functions and related transformations are discussed by Cohen (1995).

Figure C-1: Random velocity model (a), wavefield snapshot simulated in this model by acoustic finite-differences (b), and its 2D pseudo Wigner distribution function (c).
Isotropic angle-domain elastic reverse-time migration

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ABSTRACT
Multicomponent data are not usually processed with specifically designed procedures, but with procedures analogous to the ones used for single-component data. In isotropic media, the vertical and horizontal components of the data are commonly taken as proxies for the P- and S-wave modes which are imaged independently with acoustic wave equations. This procedure works only if the vertical and horizontal component accurately represent P- and S-wave modes, which is not true in general. Therefore, multicomponent images constructed with this procedure exhibit artifacts caused by the incorrect wave mode separation at the surface.
An alternative procedure for elastic imaging uses the full vector fields for wavefield reconstruction and imaging. The wavefields are reconstructed using the multicomponent data as a boundary condition for a numerical solution to the elastic wave equation. The key component for wavefield migration is the imaging condition that evaluates the match between wavefields reconstructed from sources and receivers. For vector wavefields, a simple component-by-component cross-correlation between two wavefields leads to artifacts caused by crosstalk between the unseparated wave modes. An alternative method is to separate elastic wavefields after reconstruction in the subsurface and implement the imaging condition as cross-correlation of pure wave modes instead of the Cartesian components of the displacement wavefield. This approach leads to images that are easier to interpret, since they describe reflectivity of specified wave modes at interfaces of physical properties.
As for imaging with acoustic wavefields, the elastic imaging condition can be formulated conventionally (cross-correlation with zero lag in space and time), as well as extended to non-zero space and time lags. The elastic images produced by an extended imaging condition can be used for angle decomposition of primary (PP or SS) and converted (PS or SP) reflectivity. Angle gathers constructed with this procedure have applications for migration velocity analysis and amplitude versus angle analysis.

INTRODUCTION
Seismic processing is usually based on acoustic wave equations, which assume that the Earth represents a liquid that propagates only compressional waves. Although useful in practice, this assumption is not theoretically valid. Earth materials allow for both compressional and shear wave propagation in the subsurface. Shear waves, either generated at the source or converted from compressional waves at various interfaces in the subsurface, are detected by multicomponent receivers. Shear waves are usually stronger at large incidence and reflection angles, often corresponding to large offsets. However, for complex geological structures near the surface, shear waves can be quite significant even at small offsets. Conventional single-component imaging ignores shear
wave modes, which often leads to incorrect characterization of wave-propagation, incomplete illumination of the subsurface and poor amplitude characterization.

Even when multicomponent data are used for imaging, they are usually not processed with specifically designed procedures. Instead, those data are processed with ad-hoc procedures borrowed from acoustic wave equation imaging algorithms. For isotropic media, a typical assumption is that the recorded vertical and in plane horizontal components are good approximations for the P- and S-wave modes, respectively, which can be imaged independently. This assumption is not always correct, leading to errors and noise in the images, since P- and S-wave modes are normally mixed on all recorded components. Also, since P and S modes are mixed on all components, true-amplitude imaging is questionable no matter how accurate the wavefield reconstruction and imaging condition are.

Multicomponent imaging has long been an active research area for exploration geophysicists. Techniques proposed in the literature perform imaging by using time extrapolation, e.g. by Kirchhoff migration (Kuo and Dai 1984a; Hokstad 2000) and reverse-time migration (Whitmore 1995; Chang and McMechan 1986, 1994) adapted for multicomponent data. The reason for working in the time domain, as opposed to the depth domain, is that the coupling of displacements in different directions in elastic wave equations makes it difficult to derive a dispersion relation that can be used to extrapolate wavefields in depth (Clayton and Brown 1979; Clayton 1981).

Early attempts at multicomponent imaging used the Kirchhoff framework and involve wave-mode separation on the surface prior to wave-equation imaging (Wapenaar et al. 1987; Wapenaar and Haimé 1990). Kuo and Dai (1984b) perform shot-profile elastic Kirchhoff migration, and Hokstad (2000) performs survey-sinking elastic Kirchhoff migration. Although these techniques represent different migration procedures, they compute travel-times for both PP and PS reflections, and sum data along these travel time trajectories. This approach is equivalent to distinguishing between PP reflection and PS reflections, and applying acoustic Kirchhoff migration for each mode separately. When geology is complex, the elastic Kirchhoff migration technique suffers from drawbacks similar to those of acoustic Kirchhoff migration because ray theory breaks down (Gray et al. 2001).

There are two main difficulties with independently imaging P and S wave modes separated on the surface. The first is that conventional elastic migration techniques either consider vertical and horizontal components of recorded data as P and S modes, which is not always accurate, or separate these wave modes on the recording surface using approximations, e.g. polarization (Pestana et al. 1989) or elastic potentials (Etgen 1988; Zhe and Greenhalgh 1997) or wavefield extrapolation in the vicinity of the acquisition surface (Wapenaar et al. 1990; Admundsen and Reitan 1995). Other elastic reverse time migration techniques do not separate wave modes on the surface and reconstruct vector fields, but use imaging conditions based on ray tracing (Chang and McMechan 1986, 1994) that are not always robust in complex geology. The second difficulty is that images produced independently from P and S modes are hard to interpret together, since often they do not line-up consistently, thus
requiring image post processing, e.g. by manual or automatic registration of the images \cite{Gaiser1996, Fomel2003, Nickel2004}.

We advocate an alternative procedure for imaging elastic wavefield data. Instead of separating wavefields into scalar wave modes on the acquisition surface followed by scalar imaging of each mode independently, we use the entire vector wavefields for wavefield reconstruction and imaging. The vector wavefields are reconstructed using the multicomponent vector data as boundary conditions for a numerical solution to the elastic wave equation. The key component of such a migration procedure is the imaging condition which evaluates the match between wavefields reconstructed from the source and receiver. For vector wavefields, a simple component-by-component cross-correlation between the two wavefields leads to artifacts caused by crosstalk between the unseparated wave modes, i.e. all P and S modes from the source wavefield correlate with all P and S modes from the receiver wavefield. This problem can be alleviated by using separated elastic wavefields, with the imaging condition implemented as cross-correlation of wave modes instead of cross-correlation of the Cartesian components of the wavefield. This approach leads to images that are cleaner and easier to interpret since they represent reflections of single wave modes at interfaces of physical properties.

As for imaging with acoustic wavefields, the elastic imaging condition can be formulated conventionally (cross-correlation with zero lag in space and time), as well as extended to non-zero space lags. The elastic images produced by extended imaging condition can be used for angle decomposition of PP and PS reflectivity. Angle gathers have many applications, including migration velocity analysis (MVA) and amplitude versus angle (AVA) analysis.

The advantage of imaging with multicomponent seismic data is that the physics of wave propagation is better represented, and resulting seismic images more accurately characterize the subsurface. Multicomponent images have many applications. For example they can be used to provide reflection images where the P-wave reflectivity is small, image through gas clouds where the P-wave signal is attenuated, validate bright spot reflections and provide parameter estimation for this media, Poisson’s ratio estimates, and detect fractures through shear-wave splitting for anisotropic media \cite{Li1998, Zhu1999, Knapp2001, Gaiser2001, Stewart2003, Simmons2003}. Assuming no attenuation in the subsurface, converted wave images also have higher resolution than pure-mode images in shallow part of sections, because S-waves have shorter wavelengths than P-waves. Modeling and migrating multicomponent data with elastic migration algorithms enables us to make full use of information provided by elastic data and correctly position geologic structures.

This paper presents a method for angle-domain imaging of elastic wavefield data using reverse-time migration (RTM). In order to limit the scope of our paper, we ignore several practical issues related to data acquisition and pre-processing for wave-equation migration. For example, our methodology ignores the presence of surface waves, e.g. Rayleigh and Love waves, the relatively poor spatial sampling when
imaging with multicomponent elastic data, e.g. for OBC acquisition, the presence of anisotropy in the subsurface and all amplitude considerations related to the directionality of the seismic source. All these issues are important for elastic imaging and need to be part of a practical data processing application. We restrict in this paper our attention to the problem of wave-mode separation after wavefield extrapolation and angle-decomposition after the imaging condition. These issues are addressed in more detail in a later section of the paper.

We begin by summarizing wavefield imaging methodology, focusing on reverse-time migration for wavefield multicomponent migration. Then, we describe different options for wavefield multicomponent imaging conditions, e.g. based on vector displacements and vector potentials. Finally, we describe the application of extended imaging conditions to multicomponent data and corresponding angle decomposition. We illustrate the wavefield imaging techniques using data simulated from the Marmousi II model (Martin et al., 2002).

WAVEFIELD IMAGING

Seismic imaging is based on numerical solutions to wave equations, which can be classified into ray-based (integral) solutions and wavefield-based (differential) solutions. Kirchhoff migration is a typical ray-based imaging procedure which is computationally efficient but often fails in areas of complex geology, such as sub-salt, because the wavefield is severely distorted by lateral velocity variations leading to complex multi-pathing. Wavefield imaging works better for complex geology, but is more expensive than Kirchhoff migration. Depending on computational time constraints and available resources, different levels of approximation are applied to accelerate imaging, i.e. one-way vs. two-way, acoustic vs. elastic, isotropic vs. anisotropic, etc.

Despite the complexity of various types of wavefield migration algorithms, any wavefield imaging method can be separated into two parts: wavefield reconstruction followed by the application of an imaging condition. For prestack depth migration, source and receiver wavefields have to be reconstructed at all locations in the subsurface. The wavefield reconstruction can be carried out using extrapolation in either depth or time, and with different modeling approaches, such as finite-differences (Dablain, 1986; Alford et al., 1974), finite-elements (Bolt and Smith, 1976), or spectral methods (Seriani and Priolo, 1991; Seriani et al., 1992; Dai and Cheadle, 1996). After reconstructing wavefields with the recorded data as boundary conditions into the subsurface, an imaging condition must be applied at all locations in the subsurface in order to obtain a seismic image. The simplest types of imaging conditions are based on cross-correlation or deconvolution of the reconstructed wavefields (Claerbout, 1971). These imaging conditions can be implemented in the time or frequency domain depending on the domain in which wavefields have been reconstructed. Here, we concentrate on reverse-time migration with wavefield reconstruction and imaging condition implemented in the time domain.
Reverse-time migration

Reverse-time migration reconstructs the source wavefield forward in time and the receiver wavefield backward in time. It then applies an imaging condition to extract reflectivity information out of the reconstructed wavefields. The advantages of reverse-time migration over other depth migration techniques are that the extrapolation in time does not involve evanescent energy, and no dip limitations exist for the imaged structures (McMechan, 1982, 1983; Whitmore, 1983; Baysal et al., 1983). Although conceptually simple, reverse-time migration has not been used extensively in practice due to its high computational cost. However, the algorithm is becoming more and more attractive to the industry because of its robustness in imaging complex geology, e.g. sub-salt (Jones et al., 2007; Boechat et al., 2007).

McMechan (1982, 1983), Whitmore (1983) and Baysal et al. (1983) first used reverse-time migration for poststack or zero-offset data. The procedure underlying poststack reverse-time migration is the following: first, reverse the recorded data in time; second, use these reversed data as sources along the recording surface to propagate the wavefields in the subsurface; third, extract the image at zero time, e.g. apply an imaging condition. The principle of poststack reverse-time migration is that the subsurface reflectors work as exploding reflectors and that the wave equation used to propagate data can be applied either forward or backward in time by simply reversing the time axis (Levin, 1984).

Chang and McMechan (1986) apply reverse-time migration to prestack data. Prestack reverse-time migration reconstructs source and receiver wavefields. The source wavefield is reconstructed forward in time, and the receiver wavefield is reconstructed backward in time. Chang and McMechan (1986, 1994) use a so called excitation-time imaging condition, where images are formed by extracting the receiver wavefield at the time taken by a wave to travel from the source to the image point. This imaging condition is a special case of the cross-correlation imaging condition of Claerbout (1971).

Elastic imaging vs. acoustic imaging

Multicomponent elastic data are often recorded in land or marine (ocean-bottom) seismic experiments. However, as mentioned earlier, elastic vector wavefields are not usually processed by specifically designed imaging procedures, but rather by extensions of techniques used for scalar wavefields. Thus, seismic data processing does not take full advantage of the information contained by elastic wavefields. In other words, it does not fully unravel reflections from complex geology or correctly preserve imaging amplitudes and estimate model parameters, etc.

Elastic wave propagation in an infinite homogeneous isotropic medium is charac-
terized by the wave equation (Aki and Richards 2002)

\[ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \mathbf{f} + (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} \]  

(1)

where \( \mathbf{u} \) is the vector displacement wavefield, \( t \) is time, \( \rho \) is the density, \( \mathbf{f} \) is the body source force, \( \lambda \) and \( \mu \) are the Lamé moduli. This wave equation assumes a slowly varying stiffness tensor over the imaging space. For isotropic media, one can process the elastic data either by separating wave-modes and migrating each mode using methods based on acoustic wave theory, or by migrating the whole elastic data set based on the elastic wave equation ?? . The elastic wavefield extrapolation using equation ?? is usually performed in time by Kirchhoff migration or reverse-time migration.

Acoustic Kirchhoff migration is based on diffraction summation, which accumulates the data along diffraction curves in the data space and maps them onto the image space. For multicomponent elastic data, Kuo and Dai (1984a) discuss Kirchhoff migration for shot-record data. Here, identified PP and PS reflections can be migrated by computing source and receiver traveltimes using P-wave velocity for the source rays, and P- and S-wave velocities for the receiver rays. Hokstad (2000) performs multicomponent anisotropic Kirchhoff migration for multi-shot, multi-receiver experiments, where pure-mode and converted mode images are obtained by redatuming visco-elastic vector wavefields and application of a survey-sinking imaging condition to the reconstructed vector wavefields. The wavefield separation is effectively done by the Kirchhoff integral which handles both P- and S-waves, although this technique fails in areas of complex geology where ray theory breaks down.

Elastic reverse-time migration has the same two components as acoustic reverse-time migration: reconstruction of source and receiver wavefield and application of an imaging condition. The source and receiver wavefields are reconstructed by forward and backward propagation in time with various modeling approaches. For acoustic reverse-time migration, wavefield reconstruction is done with the acoustic wave-equation using the recorded scalar data as boundary condition. In contrast, for elastic reverse-time migration, wavefield reconstruction is done with the elastic wave-equation using the recorded vector data as boundary condition.

Since pure-mode and converted-mode reflections are mixed on all components of recorded data, images produced with reconstructed elastic wavefields are characterized by crosstalk due to the interference of various wave modes. In order to obtain images with clear physical meanings, most imaging conditions separate wave modes . There are two potential approaches to separate wavefields and image elastic seismic wavefields. The first option is to separate P and S modes on the acquisition surface from the recorded elastic wavefields. This procedure involves either approximations for the propagation path and polarization direction of the recorded data, or reconstruction of the seismic wavefields in the vicinity of the acquisition surface by a numerical solution of the elastic wave equation, followed by wavefield separation of scalar and vector potentials using Helmholtz decomposition (Etgen 1988; Zhe...
An alternative data decomposition using P and S potentials reconstructs wavefields in the subsurface using the elastic wave equation, then decomposes the wavefields into P- and S-wave modes. This is followed by forward extrapolation of the separated wavefields back to the surface using the acoustic wave equation with the appropriate propagation velocity for the various wave modes (Sun et al., 2006) by conventional procedures used for scalar wavefields.

The second option is to extrapolate wavefields in the subsurface using a numerical solution to the elastic wave equation and then apply an imaging condition that extracts reflectivity information from the source and receiver wavefields. In the case where extrapolation is implemented by finite-difference methods (Chang and McMechan, 1986, 1994), this procedure is known as elastic reverse-time migration, and is conceptually similar to acoustic reverse-time migration (Baysal et al., 1983), which is more frequently used in seismic imaging.

Many imaging conditions can be used for reverse-time migration. Elastic imaging conditions are more complex than acoustic imaging conditions because both source and receiver wavefields are vector fields. Different elastic imaging conditions have been proposed for extracting reflectivity information from reconstructed elastic wavefields. Hokstad et al. (1998) use elastic reverse-time migration with Lamé potential methods. Chang and McMechan (1986) use the excitation-time imaging condition which extracts reflectivity information from extrapolated wavefields at travel times from the source to image positions computed by ray tracing, etc. Ultimately, these imaging conditions represent special cases of a more general type of imaging condition that involves time cross-correlation or deconvolution of source and receiver wavefields at every location in the subsurface.

CONVENTIONAL ELASTIC IMAGING CONDITIONS

For vector elastic wavefields, the cross-correlation imaging condition needs to be implemented on all components of the displacement field. The problem with this type of imaging condition is that the source and receiver wavefields contain a mix of P- and S-wave modes which cross-correlate independently, thus hampering interpretation of migrated images. An alternative to this type of imaging performs wavefield separation of scalar and vector potentials after wavefield reconstruction in the imaging volume, but prior to the imaging condition and then cross-correlate pure modes from the source and receiver wavefields, as suggested by Dellinger and Etgen (1990) and illustrated by Cunha Filho (1992).

Imaging with scalar wavefields

As mentioned earlier, assuming single scattering in the Earth (Born approximation), a conventional imaging procedure consists of two components: wavefield extrapolation and imaging. Wavefield extrapolation is used to reconstruct in the imaging
volume the seismic wavefield using the recorded data on the acquisition surface as a boundary condition, and imaging is used to extract reflectivity information from the extrapolated source and receiver wavefields.

Assuming scalar recorded data, wavefield extrapolation using a scalar wave equation reconstructs scalar source and receiver wavefields, $u_s(\mathbf{x}, t)$ and $u_r(\mathbf{x}, t)$, at every location $\mathbf{x}$ in the subsurface. Using the extrapolated scalar wavefields, a conventional imaging condition \cite{Claerbout1985} can be implemented as cross-correlation at zero-lag time:

$$I(\mathbf{x}) = \int u_s(\mathbf{x}, t) u_r(\mathbf{x}, t) \, dt.$$  \hfill (2)

Here, $I(\mathbf{x})$ denotes a scalar image obtained from scalar wavefields $u_s(\mathbf{x}, t)$ and $u_r(\mathbf{x}, t)$, $\mathbf{x} = \{x, y, z\}$ represent Cartesian space coordinates, and $t$ represents time.

### Imaging with vector displacements

Assuming vector recorded data, wavefield extrapolation using a vector wave equation reconstructs source and receiver wavefields $\mathbf{u}_s(\mathbf{x}, t)$ and $\mathbf{u}_r(\mathbf{x}, t)$ at every location $\mathbf{x}$ in the subsurface. Here, $\mathbf{u}_s$ and $\mathbf{u}_r$ represent displacement fields reconstructed from data recorded by multicomponent geophones at the surface boundary. Using the vector extrapolated wavefields $\mathbf{u}_s = \{u_{sx}, u_{sy}, u_{sz}\}$ and $\mathbf{u}_r = \{u_{rx}, u_{ry}, u_{rz}\}$, an imaging condition can be formulated as a straightforward extension of equation 2 by cross-correlating all combinations of components of the source and receiver wavefields. Such an imaging condition for vector displacements can be formulated mathematically as

$$I_{ij}(\mathbf{x}) = \int u_{si}(\mathbf{x}, t) u_{rj}(\mathbf{x}, t) \, dt,$$  \hfill (3)

where the quantities $u_i$ and $u_j$ stand for the Cartesian components $x, y, z$ of the vector source and receiver wavefields, $\mathbf{u}(\mathbf{x}, t)$. For example, $I_{zz}(\mathbf{x})$ represents the image component produced by cross-correlating of the $z$ components of the source and receiver wavefields, and $I_{zx}(\mathbf{x})$ represents the image component produced by cross-correlating of the $z$ component of the source wavefield with the $x$ component of the receiver wavefield, etc. In general, an image produced with this procedure has nine components at every location in space.

The main drawback of applying this type of imaging condition is that the wavefield used for imaging contains a combination of P- and S-wave modes. Those wavefield vectors interfere with one-another in the imaging condition, since the P and S components are not separated in the extrapolated wavefields. The crosstalk between various components of the wavefield creates artifacts and makes it difficult to interpret the images in terms of pure wave modes, e.g. PP or PS reflections. This situation is similar to the case of imaging with acoustic data contaminated by multiples or other types of coherent noise which are mapped in the subsurface using an incorrect velocity.
Imaging with scalar and vector potentials

An alternative to the elastic imaging condition from equation ?? is to separate the extrapolated wavefield into P and S potentials after extrapolation and image using cross-correlations of the vector and scalar potentials [Dellinger and Etgen, 1990]. Separation of scalar and vector potentials can be achieved by Helmholtz decomposition, which is applicable to any vector field \( \mathbf{u}(x,t) \):

\[
\mathbf{u} = \nabla \Phi + \nabla \times \Psi ,
\]

(4)

where \( \Phi (x,t) \) represents the scalar potential of the wavefield \( \mathbf{u}(x,t) \) and \( \Psi (x,t) \) represents the vector potential of the wavefield \( \mathbf{u}(x,t) \), and \( \nabla \cdot \Psi = 0 \). For isotropic elastic wavefields, equation ?? is not used directly in practice, but the scalar and vector components are obtained indirectly by the application of the divergence (\( \nabla \cdot \)) and curl (\( \nabla \times \)) operators to the extrapolated elastic wavefield \( \mathbf{u}(x,t) \):

\[
P = \nabla \cdot \mathbf{u} = \nabla^2 \Phi ,
\]

(5)

\[
S = \nabla \times \mathbf{u} = -\nabla^2 \Psi .
\]

(6)

For isotropic elastic fields far from the source, quantities \( P \) and \( S \) describe compressional and transverse components of the wavefield, respectively [Aki and Richards, 2002]. In 2D, the quantity \( S \) corresponds to SV waves that are polarized in the propagation plane.

Using the separated scalar and vector components, we can formulate an imaging condition that combines various incident and reflected wave modes. The imaging condition for vector potentials can be formulated mathematically as

\[
I_{ij}(x) = \int \alpha_{si}(x,t) \alpha_{rj}(x,t) \, dt ,
\]

(7)

where the quantities \( \alpha_i \) and \( \alpha_j \) stand for the various wave modes \( \alpha = \{P,S\} \) of the vector source and receiver wavefields \( \mathbf{u}(x,t) \). For example, \( I_{PP}(x) \) represents the image component produced by cross-correlating of the \( P \) wave mode of the source and receiver wavefields, and \( I_{PS}(x) \) represents the image component produced by cross-correlating of the \( P \) wave mode of the source wavefield with the \( S \) wave-mode of the receiver wavefield, etc. In isotropic media, an image produced with this procedure has four independent components at every location in space, similar to the image produced by the cross-correlation of the various Cartesian components of the vector displacements. However, in this case, the images correspond to various combinations of incident P or S and reflected P- or S-waves, thus having clear physical meaning and being easier to interpret for physical properties.

EXTENDED ELASTIC IMAGING CONDITIONS

The conventional imaging condition from equation ?? discussed in the preceding section uses zero space- and time-lags of the cross-correlation between the source and
receiver wavefields. This imaging condition represents a special case of a more general form of an extended imaging condition (Sava and Fomel, 2006b)

\[ I(x, \lambda, \tau) = \int u_s(x - \lambda, t - \tau) u_r(x + \lambda, t + \tau) dt, \quad (8) \]

where \( \lambda = \{\lambda_x, \lambda_y, \lambda_z\} \) and \( t \) stand for cross-correlation lags in space and time, respectively. The imaging condition from equation 8 is equivalent to the extended imaging condition from equation ?? for \( \lambda = 0 \) and \( t = 0 \).

The extended imaging condition has two main uses. First, it characterizes wavefield reconstruction errors, since for incorrectly reconstructed wavefields, the cross-correlation energy does not focus completely at zero lags in space and time. Sources of wavefield reconstruction errors include inaccurate numeric solutions to the wave-equation, inaccurate models used for wavefield reconstruction, inadequate wavefield sampling on the acquisition surface, and uneven illumination of the subsurface. Typically, all these causes of inaccurate wavefield reconstruction occur simultaneously and it is difficult to separate them after imaging. Second, assuming accurate wavefield reconstruction, the extended imaging condition can be used for angle decomposition. This leads to representations of reflectivity as a function of angles of incidence and reflection at all points in the imaged volume (Sava and Fomel, 2003). Here, we assume that wavefield reconstruction is accurate and concentrate on further extensions of the imaging condition, such as angle decomposition.

**Imaging with vector displacements**

For imaging with vector wavefields, the extended imaging condition from equation 8 can be applied directly to the various components of the reconstructed source and receiver wavefields, similar to the conventional imaging procedure described in the preceding section. Therefore, an extended image constructed from vector displacement wavefields is

\[ I_{ij}(x, \lambda, \tau) = \int u_{si}(x - \lambda, t - \tau) u_{rj}(x + \lambda, t + \tau) dt, \quad (9) \]

where the quantities \( u_{si} \) and \( u_{rj} \) stand for the Cartesian components \( x, y, z \) of the vector source and receiver wavefields, and \( \lambda \) and \( t \) stand for cross-correlation lags in space and time, respectively. This imaging condition suffers from the same drawbacks described for the similar conventional imaging condition applied to the Cartesian components of the reconstructed wavefields, i.e. crosstalk between the unseparated wave modes.

**Imaging with scalar and vector potentials**

An extended imaging condition can also be designed for elastic wavefields decomposed in scalar and vector potentials, similar to the conventional imaging procedure
Figure 1: Local wave vectors of the converted wave at a common image point location in 3D. The plot shows the conversion in the reflection plane in 2D. $p_s$, $p_r$, $p_x$, and $p_\lambda$ are ray parameter vectors for the source ray, receiver ray, and combinations of the two. The length of the incidence and reflection wave vectors are inversely proportional to the incidence and reflection wave velocity, respectively. Vector $n$ is the normal of the reflector. By definition, $p_x = p_r - p_s$ and $p_\lambda = p_r + p_s$. 

 geo2008IsotropicAngleDomainElasticRTM/XFig cwang
described in the preceding section. Therefore, an extended image constructed from scalar and vector potentials is

\[ I_{ij}(\mathbf{x}, \lambda, \tau) = \int \alpha_{si}(\mathbf{x} - \lambda, t - \tau) \alpha_{rj}(\mathbf{x} + \lambda, t + \tau) dt , \]

where the quantities \( \alpha_{si} \) and \( \alpha_{rj} \) stand for the various wave modes \( \alpha = \{P, S\} \) of the source and receiver wavefields, and \( \lambda \) and \( t \) stand for cross-correlation lags in space and time, respectively.

### Angle Decomposition

![Figure 2](geo2008IsotropicAngleDomainElasticRTM/XFig cwgat)

Figure 2: (a) Model showing one shot over multiple reflectors dipping at 0\(^\circ\), 15\(^\circ\), 30\(^\circ\), 45\(^\circ\) and 60\(^\circ\). The vertical dashed line shows a CIG location. Incidence ray is vertically down and P to S conversions are marked by arrowed lines pointing away from reflectors. (b) Converted wave angle gather obtained from algorithm described by Sava and Fomel (2006a). Notice that converted wave angles are always smaller than incidence angles (in this case, the dips of the reflectors) except for normal incidence.

As indicated earlier, the main uses of images constructed using extended imaging conditions are migration velocity analysis (MVA) and amplitude versus angle analysis (AVA). Such analyses, however, require that the images be decomposed in components corresponding to various angles of incidence. Angle decomposition takes different forms corresponding to the type of wavefields involved in imaging. Thus, we can distinguish angle decomposition for scalar (acoustic) wavefields and angle decomposition for vector (elastic) wavefields.
Scalar wavefields

For the case of imaging with the acoustic wave equation, the reflection angle corresponding to incidence and reflection of P-wave mode can be constructed after imaging, using mapping based on the relation (Sava and Fomel, 2005)

\[ \tan \theta_a = \frac{|k_\lambda|}{|k_x|}, \tag{11} \]

where \( \theta_a \) is the incidence angle, and \( k_x = k_r - k_s \) and \( k_\lambda = k_r + k_s \) are defined using the source and receiver wavenumbers, \( k_s \) and \( k_r \). The information required for decomposition of the reconstructed wavefields as a function of wavenumbers \( k_x \) and \( k_\lambda \) is readily available in the images \( I(x, \lambda, \tau) \) constructed by extended imaging conditions equations ?? or ??.

After angle decomposition, the image \( I(x, \theta, \phi) \) represents a mapping of the image \( I(x, \lambda, \tau) \) from offsets to angles. In other words, all information for characterizing angle-dependent reflectivity is already available in the image obtained by the extended imaging conditions.

Vector wavefields

A similar approach can be used for decomposition of the reflectivity as a function of incidence and reflection angles for elastic wavefields imaged with extended imaging conditions equations ?? or ??.

The angle \( \theta_e \) characterizing the average angle between incidence and reflected rays can be computed using the expression (Sava and Fomel, 2005)

\[ \tan^2 \theta_e = \frac{(1 + \gamma)^2 |k_\lambda|^2 - (1 - \gamma)^2 |k_x|^2}{(1 + \gamma)^2 |k_x|^2 - (1 - \gamma)^2 |k_\lambda|^2}, \tag{12} \]

where \( \gamma \) is the velocity ratio of the incident and reflected waves, e.g. \( V_P/V_S \) ratio for incident P mode and reflected S mode. Figure ?? shows the schematic and the notations used in equation ??, where \( |p_x| = |k_x|/\omega, \ |p_\lambda| = |k_\lambda|/\omega, \) and \( \omega \) is the angular frequency at the imaging location \( x \). The angle decomposition equation ?? is designed for PS reflections and reduces to equation ?? for PP reflections when \( \gamma = 1 \).

Angle decomposition using equation ?? requires computation of an extended imaging condition with 3D space lags \( (\lambda_x, \lambda_y, \lambda_z) \), which is computationally costly. Faster computation can be done if we avoid computing the vertical lag \( \lambda_z \), in which case the angle decomposition can be done using the expression (Sava and Fomel, 2005):

\[ \tan \theta_e = \frac{(1 + \gamma) (a_{\lambda_x} + b_x)}{2 \gamma k_z + \sqrt{4 \gamma^2 k_z^2 + (\gamma^2 - 1) (a_{\lambda_x} + b_x) (a_x + b_{\lambda_x})}}, \tag{13} \]

where \( a_{\lambda_x} = (1 + \gamma) k_{\lambda_x}, \ b_x = (1 + \gamma) k_x, \ b_{\lambda_x} = (1 - \gamma) k_{\lambda_x}, \) and \( b_x = (1 - \gamma) k_x \). Figure ?? shows a model of five reflectors and the extracted angle gathers for these reflectors at the location of the source. For PP reflections, they would occur in the angle gather at angles equal with the reflector slopes. However, for PS reflections, as
illustrated in Figure 2, the reflection angles are smaller than the reflector slopes, as expected.

EXAMPLES

We test the different imaging conditions discussed in the preceding sections with data simulated on a modified subset of the Marmousi II model (Martin et al., 2002). The section is chosen to be at the left side of the entire model which is relatively simple, and therefore it is easier to examine the quality of the images.

Figure 3: (a) P- and S-wave velocity models and (b) density model used for isotropic elastic wavefield modeling, where $V_P$ ranges from 1.6 to 3.2 km/s from top to bottom and $V_P/V_S = 2$, and density ranges from 1 to 2 g/cm$^3$. geo2008IsotropicAngleDomainElasticRTM/marm2oneA vp,rx
Figure 4: Elastic data simulated in model 3(a) and 3(b) with a source at $x = 6.75$ km and $z = 0.5$ km, and receivers at $z = 0.5$ km: (a) vertical component, (b) horizontal component, (c) scalar potential and (d) vector potential of the elastic wavefield. Both vertical and horizontal components, panels (a) and (b), contain a mix of P and S modes, as seen by comparison with panels (c) and (d).
Imaging with vector displacements

Consider the images obtained for the model depicted in Figures 3(a) and 3(b). Figure 3(a) depicts the P-wave velocity (smooth function between 1.6 – 3.2 km/s), and Figure 3(b) shows the density (variable between 1 – 2 g/cm³). The S-wave velocity is a scaled version of the P-wave velocity with $V_P/V_S = 2$. We use a smooth velocity background for both modeling and migration. We use density discontinuities to generate reflections in modeling, but use a constant density in migration. The smooth velocity background for both modeling and migration is used to avoid back-scattering during wavefield reconstruction. The elastic data, Figures 4(a) and 4(b), are simulated using a space-time staggered-grid finite-difference solution to the isotropic elastic wave equation (Virieux, 1984, 1986; Mora, 1987, 1988). We simulate data for a source located at position $x = 6.75$ km and $z = 0.5$ km. Since we are using an explosive source and the background velocity is smooth, the simulated wavefield is represented mainly by P-wave incident energy and the receiver wavefield is represented by a combination of P- and S-wave reflected energy. The data contain a mix of P and S modes, as can be seen by comparing the vertical and horizontal displacement components, shown in Figures 4(a) and 4(b), with the separated P and S wave modes, shown in Figures 4(c) and 4(d).

Imaging the data shown in Figures 4(a) and 4(b) using the imaging condition from equation ??, we obtain the images depicted in Figures 5(a) to 5(d). Figures 5(a) to 5(d) correspond to the cross-correlation of the $z$ and $x$ components of the source wavefield with the $z$ and $x$ components of the receiver wavefield, respectively. Since the input data do not represent separated wave modes, the images produced with the imaging condition based on vector displacements do not separate PP and PS reflectivity. Thus, the images are hard to interpret, since it is not clear what incident and reflected wave modes the reflections represent. In reality, reflections corresponding to all wave modes are present in all panels.

Imaging with scalar and vector potentials

Consider the images (Figures 6) obtained using the imaging condition from equation ?? applied to the data (Figures 4(a) and 4(b)) from the preceding example. Because we used an explosive source for our simulation, the source wavefield contains mostly P-wave energy, while the receiver wavefield contains P- and S-wave mode energy. Helmholtz decomposition after extrapolation but prior to imaging isolates P and S wavefield components. Therefore, migration produces images of reflectivity corresponding to PP and PS reflections, Figures 6(a) and 6(b), but not reflectivity corresponding to SP or SS reflections, Figures 6(c) and 6(d). The illumination regions are different between PP and PS images, due to different illumination angles of the two propagation modes for the given acquisition geometry. The PS image, Figure 6(b), also shows the usual polarity reversal for positive and negative angles of incidence measured relative to the reflector normal. By comparing Figures 6(a)
and [6(b)] with Figures 5(a) and 5(b) it is apparent that the crosstalk in the images obtained from displacement-based imaging condition is more prominent than the one obtained from potential-based imaging conditions, especially in Figure 5(a). Furthermore, the polarity in Figure 5(b) normally taken as the PS image, does not reverse polarity at normal incidence, which is not correct either.

**Angle decomposition**

The images shown in the preceding subsection correspond to the conventional imaging conditions from equations ?? and ?? . We can construct other images using the extended imaging conditions from equations ?? and ?? , which can be used for angle decomposition after imaging. Then, we can use equation ?? to compute angle gathers from horizontal space cross-correlation lags.

Figures ?? and ?? together with Figures ?? and ?? show, respectively, the PP and PS horizontal lags and angle gathers for the common image gather (CIG) location in the middle of the reflectivity model, given a single source at \( x = 6.75 \) km and \( z = 0.5 \) km. PP and PS horizontal lags are lines dipping at angles that are equal to the incidence angles (real incidence angles for PP reflection and average of incidence and reflection angles for PS reflection) at the CIG location. PP angles are larger than PS angles at all reflectors, as illustrated on the simple synthetic example shown in Figure 2.

Figures 8(a) and 8(c) together with Figures 8(b) and 8(d) show, respectively, the PP and PS horizontal lags and angle gathers for the same CIG location, given many sources from \( x = 5.5 \) to 7.5 km and \( z = 0.5 \) km. The horizontal space cross-correlation lags are focused around \( \lambda = 0 \), which justifies the use of conventional imaging condition extracting the cross-correlation of the source and receiver wavefields at zero lag in space and time. Thus, the zero lag of the images obtained by extended imaging condition represent the image at the particular CIG location. The PP and PS gathers for many sources are flat, since the migration was done with correct migration velocity. The PS angle gather, depicted in Figure 8(d), shows a polarity reversal at \( \theta = 0 \) as expected.

**DISCUSSION**

Our presentation of the angle-domain reverse-time migration method outlined in the preceding sections deliberately ignores several practical challenges in order to maintain the focus of this paper to the actual elastic imaging condition. However, for completeness, we would like to briefly mention several complementary issues that need to be addressed in conjunction with the imaging condition in order to design a practical method for elastic reverse-time migration.

First, reconstruction of the receiver wavefield requires that the multicomponent
recorded data be injected into the model in reverse-time. In other words, the recorded
data act as a displacement sources at receiver positions. In elastic materials, displace-
ment sources trigger both compressional and transverse wave modes, no matter what
portion of the recorded elastic wavefield is used as a source. For example, injecting
a recorded compressional mode triggers both a compressional (physical) mode and
a transverse (non-physical) mode in the subsurface. Both modes propagate in the
subsurface and might correlate with wave modes from the source side. There are sev-
eral ways to address this problem, such as by imaging in the angle-domain where the
non-physical modes appear as events with non-flat moveout. We can make an analogy
between those non-physical waves and multiples that also lead to non-flat events in
the angle-domain. Thus, the source injection artifacts might be eliminated by filter-
ing the migrated images in the angle domain, similar to the technique employed by
Sava and Guitton (2005) for suppressing multiples after imaging.

Second, the data recorded at a free surface contain both up-going and down-going
waves. Ideally, we should use only the up-going waves as a source for reconstructing
the elastic wavefields by time-reversal. In our examples, we assume an absorbing
surface in order to avoid this additional complication and concentrate on the imag-
ing condition. However, practical implementations require directional separation of
waves at the surface (Wapenaar and Haimé, 1990; Wapenaar et al., 1990; Admundsen
and Reitan, 1995; Admundsen et al., 2001; Hou and Marfurt, 2002). Furthermore,
a free surface allows other wave modes to be generated in the process of wavefield
reconstruction using the elastic wave-equation, e.g. Rayleigh and Love waves. Al-
though those waves do not propagate deep into the model, they might interfere with
the directional wavefield separation at the surface.

Third, we suggest in this paper that angle-dependent reflectivity constructed using
extended imaging conditions might allow for elastic AVA analysis. This theoretical
possibility requires that the wavefields are correctly reconstructed in the subsurface to
account for accurate amplitude variation. For example, boundaries between regions
with different material properties need to be reasonably located in the subsurface to
generate correct mode conversions, and the radiation pattern of the source also needs
to be known. Neither one of these aspects is part of our analyses, but they represent
important considerations for practical elastic wavefield imaging.

Fourth, the wave-mode separation using divergence and curl operators, as re-
quired by Helmholtz decomposition, does not work well in elastic anisotropic me-
dia. Anisotropy requires that the separation operators take into account the local
anisotropic parameters that may vary spatially (Yan and Sava, 2009). However, we
do not discuss anisotropic wave-mode decomposition in this paper and restrict our
attention to angle-domain imaging in isotropic models.
CONCLUSIONS

We present a method for reverse-time migration with angle-domain imaging formulated for multicomponent elastic data. The method is based on the separation of elastic wavefields reconstructed in the subsurface into pure wave-modes using conventional Helmholtz decomposition. Elastic wavefields from the source and receivers are separated into pure compressional and transverse wave-modes which are then used for angle-domain imaging. The images formed using this procedure are interpretable in terms of the subsurface physical properties, for example, by analyzing the PP or PS angle-dependent reflectivity. In contrast, images formed by simple cross-correlation of Cartesian components of reconstructed elastic wavefields mix contributions from P and S reflections and are harder to interpret. Artifacts caused by back-propagating the recorded data with displacement sources are present in both types of images, although they are easier to distinguish and attenuate on the images constructed with pure elastic wave-modes separated prior to imaging.

The methodology is advantageous not only because it forms images with clearer physical meaning, but also because it is based on more accurate physics of wave propagation in elastic materials. For example, this methodology allows for wave-mode conversions in the process of wavefield reconstruction. This is in contrast with alternative methods for multicomponent imaging which separate wave-modes on the surface and then image those independently. In addition, elastic images can be formed in the angle-domain using extended imaging conditions, which offers the potential for migration velocity analysis (MVA) and amplitude versus angle (AVA) analysis.

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Figure 5: Images produced with the displacement components imaging condition from equation ???. Panels (a), (b), (c) and (d) correspond to the cross-correlation of the vertical and horizontal components of the source wavefield with the vertical and horizontal components of the receiver wavefield, respectively. Images (a) to (d) are the $zz$, $zx$, $xz$ and $xx$ components, respectively. The image corresponds to one shot at position $x = 6.75 \text{ km}$ and $z = 0.5 \text{ km}$. Receivers are located at all locations at $z = 0.5 \text{ km}$. geo2008IsotropicAngleDomainElasticRTM/marm2oneA_ieall0,ieall1,ieall2,ieall3
Figure 6: Images produced with the scalar and vector potentials imaging condition from equation ???. Panels (a), (b), (c) and (d) correspond to the cross-correlation of the P and S components of the source wavefield with the P and S components of the receiver wavefield, respectively. Images (a) to (d) are the PP, PS, SP and SS components, respectively. The image corresponds to one shot at position \( x = 6.75 \) km and \( z = 0.5 \) km. Receivers are located at all locations at \( z = 0.5 \) km. Panels (c) and (d) are blank because an explosive source was used to generate synthetic data.
Figure 7: Horizontal cross-correlation lags for (a) PP and (c) PS reflections for the model in Figures 3(a) and 3(b). The source is at $x = 6.75$ km, and the CIG is located at $x = 6.5$ km. Panels (b) and (d) depict PP and PS angle gathers decomposed from the horizontal lag gathers in panels (a) and (c), respectively. As expected, PS angles are smaller than PP angles for a particular reflector due to smaller reflection angles.
Figure 8: Horizontal cross-correlation lags for PP (a) and PS (c) reflections for the model in Figures 3(a) and 3(b). These CIGs correspond to 81 sources from $x = 5.5$ to $7.5$ km at $z = 0.5$ km. The CIG is located at $x = 6.5$ km. Panels (b) and (d) depict PP and PS angle gathers decomposed from the horizontal lag gathers in panels (a) and (c), respectively. Since the velocity used for imaging is correct, the PP and PS gathers are flat. The PP angle gathers do not change polarity at normal incidence, but the PS angle gathers change polarity at normal incidence.
Elastic wave-mode separation for TTI media

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ABSTRACT
Seismic waves propagate through the earth as a superposition of different wave-modes. Seismic imaging in areas characterized by complex geology requires techniques based on accurate reconstruction of the seismic wavefields. A crucial component of the methods in this category, collectively known as wave-equation migration, is the imaging condition which extracts information about the discontinuities of physical properties from the reconstructed wavefields at every location in space. Conventional acoustic migration techniques image a scalar wavefield representing the P wave-mode, in contrast with elastic migration techniques which image a vector wavefield representing both the P and S wave-modes. For elastic imaging, it is desirable that the reconstructed vector fields are decomposed in pure wave-modes, such that the imaging condition produces interpretable images, characterizing for example PP or PS reflectivity. In anisotropic media, wave-mode separation can be achieved by projection of the reconstructed vector fields on the polarization vectors characterizing various wave modes. For heterogeneous media, the polarization directions change with position, therefore wave-mode separation needs to be implemented using space-domain filters. For transversely isotropic media with a tilted symmetry axis (TTI), the polarization vectors depend on the elastic material parameters, including the tilt angles. Using these parameters, I separate the wave-modes by constructing nine filters corresponding to the nine Cartesian components of the three polarization directions at every grid point. Since the S polarization vectors in TI media are not defined in the singular directions, e.g. along the symmetry axes, I construct these vectors by exploiting the orthogonality between the SV and SH polarization vectors, as well as their orthogonality with the P polarization vector. This procedure allows one to separate S wave-modes which are only kinematically correct. Realistic synthetic examples show that this wave-mode separation is effective for both 2D and 3D models with high heterogeneity and strong anisotropy.

INTRODUCTION
Although acoustic migration is currently the most common seismic imaging procedure, elastic imaging, with the addition of converted-waves, has been recognized to
have potential advantages in seeing through gas-charged sediments and in structural and near surface imaging (Stewart et al., 2003). Two options are available for elastic imaging: 1) one can separate wave-modes at the surface and image with the separated PP and PS data using acoustic migration tools (Sun et al., 2004), or 2) one can extrapolate the recorded multicomponent data and image with the reconstructed elastic wavefields by applying an imaging condition to wave-modes separated in the vicinity of the image points (Yan and Sava, 2008). The first approach benefits from the simplicity of imaging with scalar waves, but it is based on the assumption that P and S modes can be successfully separated on the recording surface, which is difficult for complicated datasets. The second approach reconstructs elastic wavefields in the subsurface, thus capturing all possible wave-mode transmissions and reflections, although it increases the computational cost in elastic wavefields modeling. In addition, the elastic migration technique requires wave-mode separation before the application of an imaging condition to avoid crosstalk between different wave-modes.

In isotropic media, the P- and S-modes (shear waves do not split in isotropic media) can easily be separated by taking the divergence and curl of the elastic wavefield (Aki and Richards, 2002), and the procedure is effective in homogeneous as well as heterogeneous media. This is because, in the far field, P- and S-waves are polarized parallel and perpendicular to the wave vectors, respectively. The polarization directions of the P- and S-waves only depend on the wave propagation direction and are not altered by the medium. Therefore, the wave-mode separators are invariant with space, and divergence and curl can always be used to separate compressional (scalar) and shear (vector) wave-modes.

However, divergence and curl do not fully separate wave-modes in anisotropic media, because P- and S-waves are not polarized parallel and perpendicular to the wave vectors. Dellinger and Etgen (1990) separate wave-modes in homogeneous VTI (vertically transversely isotropic) media by projecting the vector wavefields onto the polarization vectors of each mode. In VTI media, the polarization vectors of P- and SV-waves depend on the anisotropy parameters $\epsilon$ and $\delta$ (Thomsen, 1986) and are spatially-varying when the medium is inhomogeneous. Therefore, Yan and Sava (2009) separate wave-modes in heterogeneous VTI media by filtering the wavefields with spatially varying separators in the space domain and show that separation is effective even for complex geology with high heterogeneity.

However, VTI models are suitable only for limited geological settings with horizontal layering. Many case studies have shown that TTI (tilted transversely isotropic) models better represent complex geologies like thrusts and fold belts, e.g., the Canadian Foothills (Godfrey, 1991). Using the VTI assumption to image structures characterized by TTI anisotropy introduces both kinematic and dynamical errors in migrated images. For example, Vestrum et al. (1999) and Isaac and Lawyer (1999) show that seismic structures can be mispositioned if isotropy, or even VTI anisotropy, is assumed when the medium above the imaging targets is TTI. To carry out elastic wave-equation migration for TTI models and apply the imaging condition that cross-correlates the separated wave-modes, the wave-mode separation algorithm needs to
be adapted to TTI media. For sedimentary layers bent under geological forces, TTI migration models usually incorporate locally varying tilts, and the local symmetry axes are assumed to be orthogonal to the reflectors throughout the model (C. et al., 2008; Alkhalifah and Sava, 2010). Therefore, in complex TI models, both the local anisotropy parameters $\epsilon$ and $\delta$, and the local symmetry axes with tilt $\nu$ and azimuth $\alpha$ can be space-dependent.

This technique of separation by projecting the vector wavefields onto polarization vectors has been applied only to 2D VTI models (Dellinger, 1991; Yan and Sava, 2009) and for P-mode separation for 3D VTI models (Dellinger, 1991). For 3D models, the main challenge resides in the fact that fast and slow shear modes have non-linear polarizations along symmetry-axis propagation directions. It is possible to apply the 2D separation method to 3D TTI models using the following procedure. First, project the elastic wavefields onto symmetry planes (which contains P- and SV-modes) and their orthogonal directions (which contain the SH-mode only); then separate P- and SV-modes in the symmetry planes using divergence and curl operators for isotropic media or polarization vector projection for TI media. However, this approach is difficult as wavefields are usually constructed in Cartesian coordinates and symmetry planes of the models do not align with the Cartesian coordinates. Furthermore, for heterogeneous models, the symmetry planes change spatially, which makes projection of wavefields onto symmetry planes impossible. To avoid these problems, I propose a simpler and more straightforward solution to separate wave-modes with 3D operators, which eliminates the need for projecting the wavefields onto symmetry planes. The new approach constructs shear-wave filters by exploiting the mutual orthogonality of shear modes with the P mode, whose polarization vectors are computed by solving 3D Christoffel equations.

In this chapter, I briefly review wave-mode separation for 2D VTI media and then I extend the algorithm to symmetry planes of TTI media. Then, I generalize the wave-mode separation to 3D TI media. Finally, I demonstrate wave-mode separation in 2D with homogeneous and heterogeneous examples and separation in 3D with a homogeneous TTI example.

**WAVE-MODE SEPARATION FOR 2D TI MEDIA**

**Wave-mode separation for symmetry planes of VTI media**

Dellinger and Etgen (1990) separate quasi-P and quasi-SV modes in 2D VTI media by projecting the wavefields onto the directions in which P and S modes are polarized. For example, in the wavenumber domain, one can project the wavefields onto the P-wave polarization vectors $W_P$ to obtain quasi-P ($qP$) waves:

$$\tilde{q}P = i W_P(k) \cdot \tilde{W} = i U_x \tilde{W}_x + i U_z \tilde{W}_z,$$

where $\tilde{q}P$ is the P-wave mode in the wavenumber domain, $k = \{k_x, k_z\}$ is the
wavenumber vector, $\tilde{W}$ is the elastic wavefield in the wavenumber domain, and $W_P(k)$ is the P-wave polarization vector as a function of the wavenumber $k$.

The polarization vectors $W(k)$ of plane waves for VTI media in the symmetry planes can be found by solving the Christoffel equation (Aki and Richards 2002; Tsvankin 2005): \[
\begin{bmatrix} G - \rho V^2 I \end{bmatrix} W = 0 ,
\]
where $G$ is the Christoffel matrix with $G_{ij} = c_{ijkl} n_j n_l$, in which $c_{ijkl}$ is the stiffness tensor. The vector $n = \frac{k}{|k|}$ is the unit vector orthogonal to the plane wavefront, with $n_j$ and $n_l$ being the components in the $j$ and $l$ directions, $i, j, k, l = 1, 2, 3$. The eigenvalues $V$ of this system correspond to the phase velocities of different wave-modes and are dependent on the plane wave propagation direction $k$.

For plane waves in the vertical symmetry plane of a TTI medium, since $q_P$ and $q_{SV}$ modes are decoupled from the SH-mode and polarized in the symmetry planes, one can set $n_y = 0$ and obtain

\[
\begin{bmatrix} G_{11} - \rho V^2 & G_{12} \\ G_{12} & G_{22} - \rho V^2 \end{bmatrix} \begin{bmatrix} U_x \\ U_z \end{bmatrix} = 0 ,
\]
where

\[
\begin{align*}
G_{11} &= c_{11} n_x^2 + c_{55} n_z^2 , \\
G_{12} &= (c_{13} + c_{55}) n_x n_z , \\
G_{22} &= c_{55} n_x^2 + c_{33} n_z^2 .
\end{align*}
\]

Equation 9 allows one to compute the polarization vectors $W_P = \{U_x, U_z\}$ and $W_{SV} = \{-U_z, U_x\}$ (the eigenvectors of the matrix $G$) given the stiffness tensor at every location of the medium.

Equation 9 represents the separation process for the P-mode in 2D homogeneous VTI media. To separate wave-modes for heterogeneous models, one needs to use different polarization vectors at every location of the model (Yan and Sava 2009), because the polarization vectors change spatially with medium parameters. In the space domain, an expression equivalent to equation 6 at each grid point is

\[
q_P = \nabla a \cdot W = L_x[W_x] + L_z[W_z] ,
\]
where $L [\cdot]$ indicates spatial filtering, and $L_x$ and $L_z$ are the filters to separate P waves representing the inverse Fourier transforms of $i U_x$ and $i U_z$, respectively. The terms $L_x$ and $L_z$ define the “pseudo-derivative operators” in the $x$ and $z$ directions for a VTI medium, respectively, and they change according to the material parameters, $V_{P0}$, $V_{S0}$ ($V_{P0}$ and $V_{S0}$ are the P and S velocities along the symmetry axis, respectively), $\epsilon$, and $\delta$ (Thomsen 1986).
Wave-mode separation for symmetry planes of TTI media

My separation algorithm for TTI models is similar to the approach used for VTI models. The main difference is that for VTI media, the wavefields consist of P- and SV-modes, and equations 6 and 7 can be used for separation in all vertical planes of a VTI medium. However, for TTI media, this separation only works in the plane containing the dip of the reflector, where P- and SV-waves are polarized, while other vertical planes contain SH-waves as well.

To obtain the polarization vectors for P and S modes in the symmetry planes of TTI media, one needs to solve for the Christoffel equation 9 with

\[ G_{11} = c_{11} n_x^2 + 2c_{15} n_x n_z + c_{55} n_z^2, \]  
\[ G_{12} = c_{15} n_x^2 + (c_{13} + c_{55}) n_x n_z + c_{35} n_z^2, \]  
\[ G_{22} = c_{55} n_x^2 + 2c_{35} n_x n_z + c_{33} n_z^2. \]  

Here, since the symmetry axis of the TTI medium does not align with the vertical axis \( k_z \), the TTI Christoffel matrix is different from its VTI equivalent. The stiffness tensor is determined by the parameters \( V_P^0, V_S^0, \epsilon, \delta \), and the tilt angle \( \nu \).

In anisotropic media, \( W_P \) generally deviates from the wave vector direction \( \mathbf{k} = \frac{\omega}{V} \mathbf{n} \), where \( \omega \) is the angular frequency, \( V \) is the phase vector. Figures 1(a) and 1(b) show the P-mode polarization in the wavenumber domain for a VTI medium and a TTI medium with a 30° tilt angle, respectively. The polarization vectors for the VTI medium deviate from radial directions, which represent the isotropic polarization vectors \( \mathbf{k} \). The polarization vectors of the TTI medium are rotated 30° about the origin from the vectors of the VTI medium.

Figures ?? and ?? show the components of the P-wave polarization of a VTI medium and a TTI medium with a 30° tilt angle, respectively. Figure ?? shows that the polarization vectors in Figure ?? rotated to the symmetry axis and its orthogonal direction of the TTI medium. Comparing Figures ?? and ??, we see that within the circle of radius \( \pi \) radians, the components of this TTI medium are rotated 30° from those of the VTI medium. However, note that the z and x components of the polarization vectors for the VTI medium (Figure ??) are symmetric with respect to the x and z axes, respectively; in contrast, the vectors of the TTI medium (Figure ??) are not symmetric because of the non-alignment of the TTI symmetry with the Cartesian coordinates.

To maintain continuity at the negative and positive Nyquist wavenumbers for Fourier transform to obtain space-domain filters, i.e. at \( k_x, k_z = \pm \pi \) radians, one needs to apply tapers to the vector components. For VTI media, a taper corresponding to the function [Yan and Sava, 2009]

\[ f(k) = -\frac{8 \sin (k)}{5k} + \frac{2 \sin (2k)}{5k} - \frac{8 \sin (3k)}{105k} + \frac{\sin (4k)}{140k} \]  

can be applied to the \( x \) and \( z \) components of the polarization vectors (Figure ??), where \( k \) represent the components \( k_x \) and \( k_z \) of the vector \( \mathbf{k} \). This taper ensures that
$U_x$ and $U_z$ are zero at $k_z = \pm \pi$ radians and $k_x = \pm \pi$ radians, respectively. The components $U_x$ and $U_z$ are continuous in the $z$ and $x$ directions across the Nyquist wave numbers, respectively, due to the symmetry of the VTI media. Moreover, the application of this taper transforms polarization vector components to $8^{th}$ order derivatives. If the components of the isotropic polarization vectors $\mathbf{k}$ are tapered by the function in equation 11 and then transformed to the space domain, one obtains the conventional $8^{th}$ order finite difference derivative operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$ (Yan and Sava, 2009). Therefore, the VTI separators reduce to conventional derivatives—the components of the divergence and curl operators—when the medium is isotropic.

For TTI media, due to the asymmetry of the Fourier domain derivatives (Figure ??), one needs to apply a rotational symmetric taper to the polarization vector components to obtain continuity across Nyquist wavenumbers. A simple Gaussian taper

$$g(k) = C \exp \left[ -\frac{|k|^2}{2\sigma^2} \right]$$

(12)

can be used, where $C$ is a normalizing constant. When one chooses a standard deviation of $\sigma = 1$ radian, the magnitude of this taper at $|k| = \pi$ radians is about 0.7% of the peak value, and therefore the TTI components can be safely assumed to be continuous across the Nyquist wavenumbers. Tapering the polarization vector components in Figure 2 with the function in equation 12 one obtains the plots in Figure 3. The panels in Figure 3 which exhibits circular continuity across the Nyquist wavenumbers, transform to the space-domain separators in Figure 4. The space-domain filters for TTI media is rotated from the VTI filters, also by the tilt angle $\nu$.

The value of $\sigma$ determines the size of the operators in the space domain and also affects the frequency content of the separated wave-modes. For example, Figure 5 shows the component $U_z$ and operator $L_z$ for $\sigma$ values of 0.25, 1.00, and 1.25 radians. A larger value of $\sigma$ results in more concentrated operators in the space domain and better preserved frequency of the separated wave-modes. However, one needs to ensure that the function $g(k)$ at $|k| = \pi$ radians is small enough to assume continuity of the value function across Nyquist wavenumbers. When one chooses $\sigma = 1$ radian, the TTI components can be safely assumed to be continuous across the Nyquist wavenumbers.

For heterogeneous models, I can pre-compute the polarization vectors at each grid point as a function of the $V_{P0}/V_{S0}$ ratio, the Thomsen parameters $\epsilon$ and $\delta$, and tilt angle $\nu$. I then transform the tapered polarization vector components to the space domain to obtain the spatially-varying separators $L_x$ and $L_z$. The separators for the entire model are stored and used to separate P- and S-modes from reconstructed elastic wavefields at different time steps. Thus, wavefield separation in TI media can be achieved simply by non-stationary filtering with spatially varying operators. I assume that the medium parameters vary slowly in space and that they are locally homogeneous. For complex media, the localized operators behave similarly to the
long finite difference operators used for finite difference modeling at locations where medium parameters change rapidly.

**WAVE-MODE SEPARATION FOR 3D TI MEDIA**

In order to separate all three modes—P, SV, and SH—in a 3D TI medium, one needs to construct 3D separators. Dellinger (1991) shows that P-waves can be separated from two shear modes by a straightforward extension of the 2D algorithm. Indeed, for 3D TI media, one can always obtain the P-mode by constructing P-wave separators represented by the polarization vector $W_P = \{U_x, U_y, U_z\}$ and then projecting the 3D elastic wavefields onto the vector $W_P$. The P-wave polarization vector with components $\{U_x, U_y, U_z\}$ is obtained by solving the 3D Christoffel matrix (Aki and Richards, 2002; Tsvankin, 2005):

$$
\begin{pmatrix}
G_{11} - \rho V^2 & G_{12} & G_{13} \\
G_{12} & G_{22} - \rho V^2 & G_{23} \\
G_{13} & G_{23} & G_{33} - \rho V^2
\end{pmatrix}
\begin{pmatrix}
U_x \\
U_y \\
U_z
\end{pmatrix} = 0.
$$

(13)

The notations in this equation have the same definitions as in equation 8. For TTI media, the matrix $G$ has the elements

$$G_{11} = c_{11}n_x^2 + c_{66}n_y^2 + c_{55}n_z^2 + 2c_{16}n_x n_y + 2c_{15}n_x n_z + 2c_{56}n_y n_z,$$

(14)

$$G_{22} = c_{66}n_x^2 + c_{22}n_y^2 + c_{44}n_z^2 + 2c_{26}n_x n_y + (c_{45} + c_{46})n_x n_z + 2c_{24}n_y n_z,$$

(15)

$$G_{33} = c_{55}n_x^2 + c_{44}n_y^2 + c_{33}n_z^2 + 2c_{35}n_x n_z + 2c_{34}n_y n_z,$$

(16)

$$G_{12} = c_{16}n_x^2 + c_{26}n_y^2 + c_{45}n_z^2 + (c_{12} + c_{66})n_x n_y + (c_{14} + c_{56})n_x n_z + (c_{25} + c_{46})n_y n_z,$$

(17)

$$G_{13} = c_{15}n_x^2 + c_{46}n_y^2 + c_{35}n_z^2 + (c_{14} + c_{56})n_x n_y + (c_{13} + c_{55})n_x n_z + (c_{36} + c_{45})n_y n_z,$$

(18)

$$G_{23} = c_{56}n_x^2 + c_{24}n_y^2 + c_{34}n_z^2 + (c_{25} + c_{46})n_x n_y + (c_{36} + c_{45})n_x n_z + (c_{23} + c_{44})n_y n_z.$$  

(19)

When constructing shear mode separators, one faces an additional complication: SV- and SH-waves have the same velocity along the symmetry axis of a 3D TI medium, and this singularity prevents one from obtaining polarization vectors for shear modes in this particular direction by solving the Christoffel equation (Tsvankin, 2005). In 3D TI media, the polarization of the shear modes around the singular directions are non-linear and cannot be characterized by a plane-wave solution. Consequently, constructing 3D global separators for fast and slow shear modes is difficult.

To mitigate the effects of the shear wave-mode singularity, I use the mutual orthogonality among the P, SV, and SH modes depicted in Figure 6. In this figure, vector $\mathbf{n} = \{\sin \nu \cos \alpha, \sin \nu \sin \alpha, \cos \nu\}$ represents the symmetry axis of a TTI medium, with $\nu$ and $\alpha$ being the tilt and azimuth of the symmetry axis, respectively. The
Figure 1: The polarization vectors of P-mode as a function of normalized wavenumbers $k_x$ and $k_z$ ranging from $-\pi$ radians to $+\pi$ radians, for (a) a VTI model with $V_{P0} = 3.0$ km/s, $V_{S0} = 1.5$ km/s, $\epsilon = 0.25$ and $\delta = -0.29$, and for (b) a TTI model with the same model parameters as (a) and a symmetry axis tilt $\nu = 30^\circ$. The vectors in (b) are rotated $30^\circ$ with respect to the vectors in (a) around $k_x = 0$ and $k_z = 0$. geo2009TTIModeSeparation/Matlab VTIpolar,TTIpolar
Figure 2: The $z$ and $x$ components of the polarization vectors for P-mode in the Fourier domain for (a) a VTI medium with $\epsilon = 0.25$ and $\delta = -0.29$, and for (b) a TTI medium with $\epsilon = 0.25$, $\delta = -0.29$, and $\nu = 30^\circ$. Panel (c) represents the projection of the polarization vectors shown in (b) onto the tilt axis and its orthogonal direction.
Figure 3: The wavenumber-domain vectors in Figure 2 are tapered by the function in equation 12 to avoid Nyquist discontinuity. Panel (a) corresponds to Figure 2(a), panel (b) corresponds to Figure 2(b), and panel (c) corresponds to Figure 2(e).
Figure 4: The space-domain wave-mode separators for the medium shown in Figure 1. They are the Fourier transformation of the polarization vectors shown in Figure 3. Panel (a) corresponds to Figure 3(a), panel (b) corresponds to Figure 3(b), and panel (c) corresponds to Figure 3(c). The zoomed views show $24 \times 24$ samples out of the original $64 \times 64$ samples around the center of the filters.
Figure 5: Panels (a)–(c) correspond to component $U_z$ (left) and operator $L_z$ (right) for $\sigma$ values of 0.25, 1.00, and 1.25 radians in equation 12 respectively. A larger value of $\sigma$ results in more spread components in the wavenumber domain and more concentrated operators in the space domain.
wave vector \( \mathbf{k} \) characterizes the propagation direction of a plane wave. Vectors \( \mathbf{P}, \mathbf{SV}, \mathbf{SH} \) symbolize the compressional, and fast and slow shear polarization directions, respectively. For TI media, plane waves propagate in symmetry planes, and the symmetry axis \( \mathbf{n} \) and any wave vector \( \mathbf{k} \) form a symmetry plane. For a plane wave propagating in the direction \( \mathbf{k} \), the P-wave is polarized in this symmetry plane and deviates from the vector \( \mathbf{k} \); the SV- and SH-waves are polarized perpendicular to the P-mode, in and out of the symmetry plane, respectively.

Using this mutual orthogonality among all three modes, I first obtain the SH-wave polarization vector \( W_{SH} \) by cross multiplying vectors \( \mathbf{n} \) and \( \mathbf{k} \), which ensures that the SH mode is polarized orthogonal to symmetry planes:

\[
W_{SH} = \mathbf{n} \times \mathbf{k} = \{k_z n_y - k_y n_z,
\quad k_x n_z - k_z n_x,
\quad k_y n_x - k_x n_y\}.
\]  

Then I calculate the SV polarization vector \( W_{SV} \) by cross multiplying polarization vectors P and SH modes, which ensures the orthogonality between SV and P modes and SV and SH modes:

\[
W_{SV} = W_P \times W_{SH},
= \{k_y n_x U_y - k_x n_y U_x + k_z n_z U_z - k_x n_z U_y,
\quad k_z n_y U_x - k_y n_z U_x + k_x n_x U_y - k_y n_x U_y,
\quad k_x n_z U_x - k_z n_x U_x + k_y n_y U_y - k_z n_y U_y\}.
\]  

Here, the magnitude of the P-wave polarization vectors for a certain wavenumber \( |\mathbf{k}| \) is a constant:

\[
|U_P| = \sqrt{U_x^2 + U_y^2 + U_z^2} = c.
\]  

This ensures that for a certain wavenumber, P-waves obtained by projecting the elastic wavefields onto the polarization vectors are uniformly scaled. For comparison, the magnitudes of all three modes are respectively

\[
|U_P| = c, \quad |U_{SV}| = c \sin \phi, \quad |U_{SH}| = c \sin \phi,
\]

where \( \phi \) is the polar angle of the propagating plane wave, i.e., the angle between vectors \( \mathbf{k} \) and \( \mathbf{n} \). Figure 7 shows the polarization vectors of P-, SH-, and SV-modes computed using equations \[13, 20\] and \[21\] respectively. The P-wave polarization vectors in Figure 7(a) all have the same magnitude, but the SV and SH polarization vectors in Figures 7(c) and (b) vary in magnitude. In the symmetry axis direction, they become zero. The zero amplitude of the shear modes in the symmetry axis direction is not an abrupt but a continuous change over nearby propagation angles.
Using separators represented by solutions to equation [13] and expressions [20] and [21] to filter the wavefields, I obtain separated shear modes that are scaled differently than the P-mode. For a certain wavenumber, the shear modes are scaled by \( \sin \phi \), with \( \phi \) being the polar angle, which increases from zero in the symmetry axis to unity in the orthogonal propagation directions. Therefore, the separated SV- and SH-waves have zero amplitude in the symmetry axis direction, and the amplitudes of the shear modes are just kinematically correct.

The components of the polarization vectors for P-, SV-, and SH-waves can be transformed back to the space domain to construct spatial filters for 3D heterogeneous TI media. For example, Figure 8 illustrates nine spatial filters transformed from the Cartesian components of the polarization vectors shown in Figure 7. All these filters can be spatially varying when the medium is heterogeneous. Therefore, in principle, wave-mode separation in 3D would perform well even for models that have complex structures and arbitrary tilts and azimuths of TI symmetry.

Figure 6: A schematic showing the elastic wave-modes polarization in a 3D TI medium. The three parallel planes represent the isotropy planes of the medium. The vector \( \mathbf{n} \) represents the symmetry axis, which is orthogonal to the isotropy plane. The vector \( \mathbf{k} \) is the propagation direction of a plane wave. The wave-modes P, SV, and SH are polarized in the direction P, SV, and SH, respectively. The three modes are polarized orthogonal to each other.
Figure 7: The wave-mode polarization for P-, SH-, and SV-mode for a VTI medium with parameters $V_P = 4.95$ km/s, $V_S = 2.48$ km/s, $\epsilon = 0.4$, and $\delta = 0.1$. The P-mode polarization is computed using the 3D Christoffel equation, and SV and SH polarizations are computed using Equations 21 and 20. Note that the SV- and SH-wave polarization vectors have zero amplitude in the vertical direction.
Figure 8: The separation filters $L_x$, $L_y$, and $L_z$ for the P, SV, and SH modes for a VTI medium. The corresponding wavenumber-domain polarization vectors are shown in Figure 7. Note that the filter $L_z$ for the SH mode is blank because the $z$ component of the polarization vector is zero. The zoomed views show $24 \times 24$ samples out of the original $64 \times 64$ samples around the center of the filters.
EXAMPLES

I illustrate the anisotropic wave-mode separation with a simple fold synthetic example and a more challenging model based on the elastic Marmousi II model \cite{Bourgeois1991}. I then show the wave-mode separation for a 3D TTI model.

2D TTI fold model

Consider the 2D fold model shown in Figure 9. Panels 9(a)–(f) show $V_P$, $V_S$, density, parameters $\epsilon$, $\delta$, and the local tilts $\nu$ of the model, respectively. The symmetry axis is orthogonal to the reflectors throughout the model. Figure 10 illustrates the separators obtained at different locations in the model and defined by the intersections of $x$ coordinates 0.15, 0.3, 0.45 km and $z$ coordinates 0.15, 0.3, 0.45 km, shown by the dots in Figure 10(a). Since the operators correspond to different combinations of the $V_P/V_S$ ratio and parameters $\epsilon$, $\delta$, and tilt angle $\nu$, they have different forms. However, the orientation of the operators conform to the corresponding tilts at the locations shown by the dots in Figure 10(a). For complex models, the symmetry axes vary spatially, which makes it difficult to rotate the wavefields to the local symmetry axis directions. Consequently, the elastic wavefields are reconstructed in untilted Cartesian coordinates, and when separating wave-modes, I use operators constructed in conventional Cartesian coordinates. To illustrate the relationship between the operators and the local tilts, the filters in Figure 10 are projected onto the local symmetry axes and the orthogonal directions at the filter location. As shown in Figure 4, the rotated filters (Figure ??) show a clearer relation with the tilt angle, while the non-rotated filters (Figure ??), which are used in the wave-mode separation, do not show a clear relation with the tilt angle.

Figure 7(a) shows the vertical and horizontal components of one snapshot of the simulated elastic anisotropic wavefield; Figure 11(b) shows the separation into P- and S-modes using divergence and curl operators; Figure 11(c) shows the separation into $q_P$ and $q_S$ modes using VTI filters, i.e., assuming zero tilt throughout the model; and Figure 11(d) shows the separation obtained with the TTI operators constructed using the local medium parameters with correct tilts. The isotropic separation shown in Figure 11(b) is incomplete; for example, at $x = 0.4$ km and $z = 0.1$ km, and at $x = 0.4$ km and $z = 0.35$ km, residuals for direct P and S arrivals are visible in the $q_P$ and $q_S$ panels, respectively. A comparison of Figures 11(c) and (d) indicates that the spatially-varying derivative operators with correct tilts successfully separate the elastic wavefields into $q_P$ and $q_S$ modes, while the VTI operators only work in the part of the model that is locally VTI.
Figure 9: A fold model with parameters (a) $V_{P0}$, (b) $V_{S0}$, (c) density, (d) $\epsilon$, (e) $\delta$, and (f) tilt angle $\nu$. The dots in panel (f) correspond to the locations of the anisotropic operators shown in Figure 10.
Figure 10: The TTI wave-mode separation filters projected to local symmetry axes and their orthogonal directions. Here, I use $\sigma = 1$ in equation 12 to taper the polarization vector components before the Fourier transform. The filters correspond to the intersections of $x = 0.15, 0.3, 0.45$ km and $z = 0.15, 0.3, 0.45$ km for the model shown in Figure 9. The locations of these operators are also shown by the dots in Figure 10(a).
Figure 11: (a) A snapshot of the anisotropic wavefield simulated with a vertical point displacement source at $x = 0.3$ km and $z = 0.1$ km for the model shown in Figure 9. Panels (b) to (d) are the anisotropic $q_P$ and $q_S$ modes separated using isotropic, VTI, and TTI separators, respectively. The separation is incomplete in panels (b) and (c) where the model is strongly anisotropic and where the model tilt is large, respectively. Panel (d) shows the best separation among all.
Marmousi II model

My second model (Figure 12) uses an elastic anisotropic version of the Marmousi II model (Bourgeois et al., 1991). In the modified model, \( V_{P0} \) is taken from the original model (Figure 12(a)), the \( V_{P0}/V_{S0} \) ratio ranges from 2 to 2.5, (Figure 12(b)), and the density \( \rho \) is taken from the original model (Figure 12(c)). The parameter \( \epsilon \) and \( \delta \) are derived from the density model \( \rho \) with the relations of \( \epsilon = 0.25\rho - 0.3 \) and \( \epsilon = 0.125\rho - 0.1 \), respectively. The parameter \( \epsilon \) ranges from 0.13 to 0.36 Figure 12(d) and parameter \( \delta \) ranges from 0.11 to 0.24 Figure 12(e). These anisotropy parameters are obtained by assuming linear relationships to the velocity models, and therefore, they both follow the structure of the model. Figure 12(f) represents the local dips obtained from the density model using plane wave destruction filters (Fomel, 2002). The dip model is used to simulate the wavefields and also used to construct TTI separators. A displacement source oriented at 45° to the vertical direction and located at coordinates \( x = 11 \) km and \( z = 1 \) km is used to simulate the elastic anisotropic wavefield.

Figure 13(a) presents one snapshot of the simulated elastic wavefields using the anisotropic model shown in Figure 12. Figures 13(b), (c), and (d) demonstrate the separation using conventional divergence and curl operators, VTI filters, and correct TTI filters, respectively. The VTI filters are constructed assuming zero tilt throughout the model, and the TTI filters are constructed with the dips used for modeling. As expected, the conventional divergence and curl operators fail at locations where anisotropy is strong. For example, in Figure 13(b) at coordinates \( x = 12.0 \) km and \( z = 1.0 \) km strong S-wave residual exists, and at coordinates \( x = 13.0 \) km and \( z = 1.5 \) km strong P-wave residual exists. VTI separators fail at locations where the dip is large. For example, in Figures 13(c) at coordinates \( x = 10.0 \) km and \( z = 1.2 \) km, strong S-wave residual exist. However, even for this complicated model, separation using TTI separators is effective at locations where medium parameters change rapidly.

3D TTI model

I use a homogeneous TTI model to illustrate the separation of P-, SV-, and SH-modes. The model has parameters \( V_{P0} = 3.5 \) km/s, \( V_{S0} = 1.75 \) km/s, \( \rho = 2.0 \) g/cm\(^3\), \( \epsilon = 0.4 \), \( \delta = 0.1 \), \( \gamma = 0.0 \), \( \nu = 30^\circ \), and \( \alpha = 45^\circ \). Figure 14 shows a snapshot of the elastic wavefields in the \( z \), \( x \), and \( y \) directions. A displacement source located at the center of the model and oriented at tilt 45° and azimuth 45° is used to excite the wavefield. Figure 15 shows successfully separated P-, SV-, and SH-modes. In this model, the parameter \( \gamma \), which characterizes the anisotropy of SH-mode, is set to zero so that the SH-mode propagation is isotropic. For this homogeneous model, a spherical wavefront in the SH-panel indicates successful separation of SV- and SH-modes.

Because this model is homogeneous, the separation is implemented in the wavenumber domain to reduce computation cost. For heterogeneous models, 3D non-stationary
Figure 12: Anisotropic elastic Marmousi II model with (a) $V_{P0}$, (b) $V_{S0}$, (c) density, (d) $\epsilon$, (e) $\delta$, and (f) local tilt angle $\nu$. 
geo2009TTIModeSeparation/marmousi2 vp,vs,epsilon,delta,nu
Figure 13: (a) A snapshot of the vertical and horizontal displacement wavefield simulated for model shown in Figure [12]. Panels (b) to (c) are the P- and SV-wave separation using $\nabla \cdot$ and $\nabla \times$, VTI separators and TTI separators, respectively. The separation is incomplete in panels (b) and (c) where the model is strongly anisotropic and where the model tilt is large, respectively. Panel (d) shows the best separation among all. [geo2009TTIModeSeparation/marmousi2 uA-wom,iA-wom,vA-wom,pA-wom]
filtering is necessary to separate different wave-modes. I do not perform wave-mode separation in 3D heterogeneous models because of the high computational cost, which will be discussed in more detail in the following section.

**DISCUSSION**

**Computational issues**

The separation of wave-modes for heterogeneous TI models requires non-stationary spatial filtering with large operators (operators of 50 samples in each dimension are used in this chapter), which is computationally expensive. The cost is directly proportional to the size of the model and to the size of each operator. Furthermore, in a simple implementation, the storage for the separation operators of the entire model is proportional to the size of the model and to the size of each operator. Suppose that a 3D elastic TTI model is characterized by the model parameters \(V_P^0, V_S^0,\) Thomsen parameters \(\epsilon\) and \(\delta,\) and symmetry axis tilt angle \(\nu\) and azimuth angle \(\alpha.\)

For a 3D model of \(300 \times 300 \times 300\) grid points, if one assumes that all operators have a size of \(50 \times 50 \times 50\) samples, the storage for the operators is \(300^3\) grid points \(\times \) \(50^3\) samples/independent operator \(\times 3\) independent operators/grid point \(\times 4\) Bytes/sample = 40.5 TB. This is not feasible in ordinary processing. However, since there are relatively few medium parameters, i.e., the \(V_{P0}/V_{S0}\) ratio, \(\epsilon,\) \(\delta,\) and angles \(\nu\) and \(\alpha,\) which determine the properties of the operators, one can construct a look-up table of operators as a function of these parameters, and search the appropriate operators at every location in the model when doing wave-mode separation. For example, suppose one knows that \(V_{P0}/V_{S0} \in [1.5, 2.0],\) \(\epsilon \in [0, 0.3],\) \(\delta \in [0, 0.1],\) and the symmetry axis tilt angle \(\nu \in [-90^\circ, 90^\circ]\) and azimuth angle \(\alpha \in [-180^\circ, 180^\circ],\) one can sample the \(V_{P0}/V_{S0}\) ratio at every 0.1, \(\epsilon\) and \(\delta\) at every 0.03, and the angles at every 15°. In this case, one only needs a storage of \(6 \times 10 \times 3 \times 12 \times 24\) combinations of medium parameters \(\times 50^3\) sample/independent operator \(\times 3\) independent operators/combinations of medium parameters \(\times 4\) Bytes/sample = 77 GB; this is more manageable, although it is still a large volume to store.

**S wave-mode amplitudes**

Although the procedure used in this chapter to separate S-waves into SV- and SH-modes is simple, the amplitudes of S-modes are not accurate because the S-wave separators are not normalized for any given wavenumbers. The amplitudes of S-modes obtained in this way are zero in the symmetry axis direction and they gradually increase to one in the symmetry plane.

The main problem that prevents one from constructing the 3D global shear wave separators is that the SV and SH polarization vectors are singular in the symmetry axis direction, i.e., they are not defined by the plane-wave solution of the TI elas-
Figure 14: A snapshot of the elastic wavefield in the $z$, $x$, and $y$ directions for a 3D VTI model. The model has parameters $V_{P0} = 3.5$ km/s, $V_{S0} = 1.75$ km/s, $\rho = 2.0$ g/cm$^3$, $\epsilon = 0.4$, $\delta = 0.1$, and $\gamma = 0.0$. A displacement source oriented at $45^\circ$ to the vertical direction and located at coordinates $x = 11$ km and $z = 1$ km is used to simulate the elastic anisotropic wavefield.
Figure 15: Separated P-, SV- and SH-wave-modes for the elastic wavefields shown in Figure 14. P, SV, and SH are well separated from each other.
tic wave equation. Various studies (Kieslev and Tsvankin, 1989; Tsvankin, 2005) show that S-waves excited by point forces can have non-linear polarizations in several special directions. For example, in the direction of the source, the S-wave can deviate from the linear polarization. This phenomenon exists even in isotropic media. Anisotropic velocity and amplitude variations can also cause the S-waves to be polarized non-linearly. For instance, S-wave triplication, S-wave singularities, and S-wave velocity maximum can all result in S-wave polarization anomalies. In these special directions, SV- and SH-mode polarizations are incorrectly defined by my convention. One possibility for obtaining more accurate S-wave amplitudes is to approximate the anomalous polarization with the major axes of the quasi-ellipses of the S-wave polarization, which can be obtained by incorporating the first-order term in the ray tracing method. This extension remains outside the scope of this chapter.

Although the simplified approach used in this chapter ignores the complicated polarization behavior in some wave propagation directions, it does successfully separate fast and slow shear modes kinematically. This allows one to use the separated scalar shear-modes for the subsequent imaging condition and obtain images with clear physical meaning.

CONCLUSIONS

Different wave-modes in elastic media can be separated by projecting the vector wavefields onto the polarization vectors of each mode. For heterogeneous models, it is necessary to separate wave-modes in the space domain by non-stationary filtering. I present a method for obtaining spatially-varying wave-mode separators for TI models, which can be used to separate elastic wave-modes in complex media. The method computes the components of the polarization vectors in the wavenumber domain and then transforms them to the space domain to obtain spatially-varying filters. In order for the operators to work in TI models with non-zero tilt angles, I incorporate one more parameter—the local tilt angle $\nu$—in addition to the parameters needed for the VTI operators. This kind of spatial filters can be used to separate complicated wavefields in TI models with high heterogeneity and strong anisotropy. I test the separation with synthetic models that have realistic geologic complexity. The results support the effectiveness of wave-mode separation with non-stationary filtering.

I also extend the wave-mode separation to 3D TI models. The P-mode separators can be constructed by solving the Christoffel equation for the P-wave eigenvectors with local medium parameters. The SV and SH separators are constructed using the mutual orthogonality among P, SV, and SH modes. For the three modes, there are a total number of nine separators, with three components for each mode. The separators vary according to the medium parameters $V_{P0}$, $V_{S0}$, anisotropy parameters $\epsilon$ and $\delta$ and tilt $\nu$ and azimuth $\alpha$ of the symmetry axis. The P-wave separators are constructed under no kinematic assumptions, and amplitudes of P-mode correctly characterize the plane-wave solution. Shear wave separators are constructed under kinematic assumptions, and therefore the amplitudes of shear modes are inaccurate
in the singular directions. Nevertheless, the proposed technique successfully separates fast and slow shear wavefields. The process of constructing 3D separators and separating wave-modes in 3D eliminates the step of decomposing the wavefields into symmetry planes, which only works for models with an invariant symmetry axis. Spatially-varying 3D separators have potential benefits for complex models and can be used to separate wave-modes in elastic reverse time migration (RTM) for TTI models. The spatially-varying 3D separators imply large computational and storage cost, and therefore, a more efficient separation method, such as the proposed table look-up alternative, is necessary for a successful implementation.

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Elastic wave-mode separation for VTI media

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ABSTRACT
Elastic wave propagation in anisotropic media is well represented by elastic wave equations. Modeling based on elastic wave equations characterizes both kinematics and dynamics correctly. However, because P and S modes are both propagated using elastic wave equations, there is a need to separate P and S modes to obtain clean elastic images. The separation of wave modes to P and S from isotropic elastic wavefields is typically done using Helmholtz decomposition. However, Helmholtz decomposition using conventional divergence and curl operators in anisotropic media does not give satisfactory results and leaves the different wave modes only partially separated. The separation of anisotropic wavefields requires the use of more sophisticated operators which depend on local material parameters. Anisotropic wavefield separation operators are constructed using the polarization vectors evaluated by solving the Christoffel equation at each point of the medium. These polarization vectors can be represented in the space domain as localized filtering operators, which resemble conventional derivative operators. The spatially-variable “pseudo” derivative operators perform well in heterogeneous VTI media even at places of rapid velocity/density variation. Synthetic results indicate that the operators can be used to separate wavefields for VTI media with an arbitrary degree of anisotropy.

INTRODUCTION
Wave equation migration for elastic data usually consists of two steps. The first step is wavefield reconstruction in the subsurface from data recorded at the surface. The second step is the application of an imaging condition which extracts reflectivity information from the reconstructed wavefields.

The elastic wave equation migration for multicomponent data can be implemented in two ways. The first approach is to separate recorded elastic data into compressional and transverse (P and S) modes and use the separated data for acoustic wave equation migration separately. This acoustic imaging approach to elastic waves is more frequently used, but it is fundamentally based on the assumption that P and S data can be successfully separated on the surface, which is not always true.

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The second approach is to not separate P and S modes on the surface, but to extrapolate the entire elastic wavefield at once, and then separate wave modes prior to applying an imaging condition. The reconstruction of elastic wavefields can be implemented using various techniques, including reconstruction by time reversal (RTM) (Chang and McMechan, 1986, 1994) or by Kirchhoff integral techniques (Hokstad, 2000).

The imaging condition applied to the reconstructed vector wavefields directly determines the quality of the images. Conventional crosscorrelation imaging condition does not separate the wave modes and crosscorrelates the Cartesian components of the elastic. In general, the various wave modes (P and S) are mixed on all wavefield components and cause crosstalk and image artifacts. Yan and Sava (2009) suggest using imaging conditions based on elastic potentials, which require crosscorrelation of separated modes. Potential-based imaging condition creates images that have clear physical meaning, in contrast with images obtained with Cartesian wavefield components, thus justifying the need for wave mode separation.

As the need for anisotropic imaging increases, more processing and migration are performed based on anisotropic acoustic one-way wave equations (Alkhalifah, 1998, 2000; Shan, 2006; Shan and Biondi, 2005; Fletcher et al., 2009; Fowler et al., 2010). However, much less research has been done on anisotropic elastic migration based on two-way wave equations. Elastic Kirchhoff migration (Hokstad, 2000) obtains pure-mode and converted mode images by downward continuation of elastic vector wavefields with a visco-elastic wave equation. The wavefield separation is effectively done with elastic Kirchhoff integration, which handles both P and S waves. However, Kirchhoff migration does not perform well in areas of complex geology where ray theory breaks down (Gray et al., 2001), thus requiring migration with more accurate methods, such as reverse time migration.

One of the complexities that impedes elastic wave equation anisotropic migration is the difficulty to separate anisotropic wavefields into different wave modes after reconstructing the elastic wavefields. However, the proper separation of anisotropic wave modes is as important for anisotropic elastic migration as is the separation of isotropic wave modes for isotropic elastic migration. The main difference between anisotropic and isotropic wavefield separation is that Helmholtz decomposition is only suitable for the separation of isotropic wavefields and is inadequate for anisotropic wavefields.

In this chapter, I show how to construct wavefield separators for VTI (vertical transverse isotropy) media applicable to models with spatially varying parameters. I apply these operators to anisotropic elastic wavefields and show that they successfully separate anisotropic wave modes, even for extremely anisotropic media.

The main application of this technique is in the development of elastic reverse time migration. In this case, complete wavefields containing both P and S wave modes are reconstructed from recorded data. The reconstructed wavefields are separated in pure wave modes prior to the application of a conventional crosscorrelation imaging con-
dition. I limit the scope of this chapter only to the wave-mode separation procedure in highly heterogeneous media, although the ultimate goal of this procedure is to aid elastic RTM.

**SEPARATION METHOD**

Separation of scalar and vector potentials can be achieved by Helmholtz decomposition, which is applicable to any vector field $W(x,y,z)$. By definition, the vector wavefield $W$ can be decomposed into a curl-free scalar potential $\Theta$ and a divergence-free vector potential $\Psi$ according to the relation (Aki and Richards, 2002):

$$W = \nabla \Theta + \nabla \times \Psi.$$  

Equation 1 is not used directly in practice, but the scalar and vector components are obtained indirectly by the application of the $\nabla \cdot$ and $\nabla \times$ operators to the extrapolated elastic wavefield:

$$P = \nabla \cdot W,$$  

$$S = \nabla \times W.$$  

For isotropic elastic fields far from the source, quantities $P$ and $S$ describe compressional and shear wave modes, respectively (Aki and Richards, 2002).

Equations 2 and 3 allow one to understand why $\nabla \cdot$ and $\nabla \times$ pass compressional and transverse wave modes, respectively. In the discretized space domain, one can write:

$$P = \nabla \cdot W = D_x[W_x] + D_y[W_y] + D_z[W_z],$$  

where $D_x$, $D_y$, and $D_z$ represent spatial derivatives in the $x$, $y$, and $z$ directions, respectively. Applying derivatives in the space domain is equivalent to applying finite difference filtering to the functions. Here, $D[\cdot]$ represents spatial filtering of the wavefield with finite difference operators. In the Fourier domain, one can represent the operators $D_x$, $D_y$, and $D_z$ by $i k_x$, $i k_y$, and $i k_z$, respectively; therefore, one can write an equivalent expression to equation 4 as:

$$\tilde{P} = i \mathbf{k} \cdot \tilde{W} = i k_x \tilde{W}_x + i k_y \tilde{W}_y + i k_z \tilde{W}_z,$$  

where $\mathbf{k} = \{k_x, k_y, k_z\}$ represents the wave vector, and $\tilde{W}(k_x, k_y, k_z)$ is the 3D Fourier transform of the wavefield $W(x,y,z)$. We see that in this domain, the operator $i \mathbf{k}$ essentially projects the wavefield $W$ onto the wave vector $\mathbf{k}$, which represents the polarization direction for P waves. Similarly, the operator $\nabla \times$ projects the wavefield onto the direction orthogonal to the wave vector $\mathbf{k}$, which represents the polarization direction for S waves (Dellinger and Etgen, 1990). For illustration, Figure 1(a) shows the polarization vectors of the P mode of a 2D isotropic model as a function of normalized $k_x$ and $k_z$ ranging from $-1$ to 1 cycles. The polarization vectors are radial.
Figure 1: The $q_P$ and $q_S$ polarization vectors as a function of normalized wavenumbers $k_x$ and $k_z$ ranging from $-1$ to $+1$ cycles, for (a) an isotropic model with $V_P = 3$ km/s and $V_S = 1.5$ km/s, and (b) an anisotropic (VTI) model with $V_{P0} = 3$ km/s, $V_{S0} = 1.5$ km/s, $\epsilon = 0.25$ and $\delta = -0.29$. The red arrows are the $q_P$ wave polarization vectors, and the blue arrows are the $q_S$ wave polarization vectors.

because the P waves in an isotropic medium are polarized in the same directions as the wave vectors.

Dellinger and Etgen (1990) suggest the idea that wave mode separation can be extended to anisotropic media by projecting the wavefields onto the directions in which the P and S modes are polarized. This requires that one should modify the wave separation equation [5] by projecting the wavefields onto the true polarization directions $\mathbf{U}$ to obtain quasi-P ($qP$) waves:

$$\tilde{q}P = i \mathbf{U}(k) \cdot \tilde{\mathbf{W}} = i U_x \tilde{W}_x + i U_y \tilde{W}_y + i U_z \tilde{W}_z.$$  \hspace{1cm} (6)

In anisotropic media, $\mathbf{U}(k_x, k_y, k_z)$ is different from $\mathbf{k}$, as illustrated in Figure 1(b), which shows the polarization vectors of $q_P$ wave mode for a 2D VTI anisotropic model with normalized $k_x$ and $k_z$ ranging from $-1$ to $1$ cycles. Polarization vectors are not radial because $q_P$ waves in an anisotropic medium are not polarized in the same directions as wave vectors, except in the symmetry planes ($k_z = 0$) and along the symmetry axis ($k_x = 0$).

Dellinger and Etgen (1990) demonstrate wave mode separation in the wavenumber domain using projection of the polarization vectors, as indicated in equation 6. However, for heterogeneous media, this equation is defective because the polarization vectors are spatially varying. One can write an equivalent expression to equation 6 in the space domain for each grid point as:

$$q_P = \nabla_a \cdot \mathbf{W} = L_x[W_x] + L_y[W_y] + L_z[W_z],$$  \hspace{1cm} (7)
where $L_x$, $L_y$, and $L_z$ represent the inverse Fourier transforms of $iU_x$, $iU_y$, and $iU_z$, respectively. $L[\cdot]$ represents spatial filtering of the wavefield with anisotropic separators. $L_x$, $L_y$, and $L_z$ define the pseudo derivative operators in the $x$, $y$, and $z$ directions for an anisotropic medium, respectively, and they change from location to location according to the material parameters.

We obtain the polarization vectors $U(k)$ by solving the Christoffel equation (Aki and Richards, 2002; Tsvankin, 2005):

$$\begin{bmatrix} G - \rho V^2 I \end{bmatrix} U = 0,$$

where $G$ is the Christoffel matrix $G_{ij} = c_{ijkl}n_j n_l$, in which $c_{ijkl}$ is the stiffness tensor, $n_j$ and $n_l$ are the normalized wave vector components in the $j$ and $l$ directions, $i, j, k, l = 1, 2, 3$. The parameter $V$ corresponds to the eigenvalues of the matrix $G$. The eigenvalues $V$ represent the phase velocities of different wave modes and are functions of the wave vector $k$ (corresponding to $n_j$ and $n_l$ in the matrix $G$). For plane waves propagating in any symmetry planes of a VTI medium, one can set $k_y$ to 0 and get

$$\begin{bmatrix} c_{11}k_x^2 + c_{55}k_z^2 - \rho V^2 \\ 0 \\ (c_{13} + c_{55}) k_x k_z \end{bmatrix} \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = 0. \quad (9)$$

The middle row of this matrix characterizes the SH wave polarized in the $y$ direction, and $qP$ and $qSV$ modes are uncoupled from the SH mode and are polarized in the vertical plane. The top and bottom rows of this equation allow one to compute the polarization vector $U = \{U_x, U_z\}$ (the eigenvectors of the matrix ) of P or SV wave mode given the stiffness tensor at every location of the medium.

One can extend the procedure described here to heterogeneous media by computing two different operator for each mode at every grid point. In the symmetry planes of VTI media, the operators are 2D and depend on the local values of the stiffness coefficients. For each point, I pre-compute the polarization vectors as a function of the local medium parameters, and transform them to the space domain to obtain the wave mode separators. I assume that the medium parameters vary smoothly (locally homogeneous), but even for complex media, the localized operators work in the same way as the long finite difference operators. If one represents the stiffness coefficients using Thomsen parameters (Thomsen, 1986), then the pseudo derivative operators $L_x$ and $L_z$ depend on $\epsilon$, $\delta$, $V_{P0}$ and $V_{S0}$, which can be spatially varying parameters. One can compute and store the operators for all grid points in the medium, and then use these operators to separate P and S modes from reconstructed elastic wavefields at different time steps. Thus, wavefield separation in VTI media can be achieved simply by non-stationary filtering with operators $L_x$ and $L_z$.

**OPERATOR PROPERTIES**

In this section, I discuss the properties of the anisotropic “derivative” operators, including order of accuracy, size, and compactness.
Operator orders

As I have showed in the previous section, the isotropic separation operators (divergence and curl) in equations 4 and 5 are exact in the $x$ and $k$ domains. The exact derivative operators are infinitely long series in the discretized space domain. In practice, when evaluating the derivatives numerically, one needs to take some approximations to make the operators short and computationally efficient. Usually, difference operators are evaluated at different orders of accuracy. The higher order the approximation is, the more accurate and longer the operator becomes. For example, the $2^{nd}$ order operator has coefficients $(-\frac{1}{2}, +\frac{1}{2})$, and the more accurate $4^{th}$ order operator has coefficients $(+\frac{1}{12}, -\frac{2}{3}, \frac{2}{3}, -\frac{1}{12})$ (Fornberg and Ghrist 1999).

In the wavenumber domain, for isotropic media, as shown by the black line in Figure 2(b), the exact difference operator is $ik$. Appendix A shows the $k$ domain equivalents of the $2^{nd}$, $4^{th}$, $6^{th}$, and $8^{th}$ order finite difference operators, and they are plotted in Figure 2(b). The higher order operators have responses closer to the exact operator $ik$ (black line). To obtain vertical and horizontal derivatives of different orders of accuracy, I weight the polarization vector $ik$ components $ik_x$ and $ik_z$ by the weights shown in Figure 2(c). For VTI media, similarly, I weight the anisotropic polarization vector $iU(k)$ components $iU_x$ and $iU_z$ by these same weights. The weighted vectors are then transformed back to space domain to obtain the anisotropic stencils.

Operator size and compactness

Figure 3 shows the derivative operators of $2^{nd}$, $4^{th}$, $6^{th}$, and $8^{th}$ orders in the $z$ and $x$ directions for isotropic and VTI ($\epsilon=-0.25$, $\delta=-0.29$) media. As we can see, isotropic operators become longer when the order of accuracy is higher. Anisotropic operators, however, do not change much in size. One can see that the central parts of the anisotropic operators look similar to their corresponding isotropic operators and change with the order of accuracy; while the outer parts of these anisotropic operators all look similar, and do not change much with the order of accuracy. This indicates that the central parts of the operators are determined by the order of accuracy, while the outer parts are representation of the degree of anisotropy.

Figure 4 shows anisotropic derivative operators with same order of accuracy ($8^{th}$ order in space) for three VTI media with different combinations of $\epsilon$ and $\delta$. These operators have similar central parts, but different outer parts. This result is consistent with the previous observation that the central part of an operator is determined by the order of accuracy, and the outer part is controlled by anisotropy parameters.

Figure 5(a) shows the influence of approximation to finite difference ($2^{nd}$ and $8^{th}$ order, Figures 3(h) and (b)). The “anisotropic” part (“diagonal tails”) are almost the same, and the difference comes from the central part. Figure 5(b) shows the difference between operators with different anisotropy (Figures 4(a) and (b)). The difference mainly lies in the “tails” of the operators.
Figure 2: Comparison of derivative operators of different orders of accuracy (2nd, 4th, 6th, and 8th orders in space, as well as the approximation applied in Dellinger and Etgen (1990) cosine taper) in both (a) the $x$ domain and (b) the $k$ domain. (c) Weights to apply to the components of the polarization vectors.
Figure 3: 2nd, 4th, 6th, and 8th order derivative operators for an isotropic medium ($V_P = 3$ km/s and $V_S = 1.5$ km/s) and a VTI medium ($V_{P0} = 3$ km/s, $V_{S0} = 1.5$ km/s, $\epsilon = 0.25$ and $\delta = -0.29$). The left column includes isotropic operators, and the right column includes anisotropic operators. From top to bottom are operators with increasing orders of accuracy.
A comparison between Figures [4(a) and (b)] shows that when one has large difference between $\epsilon$ and $\delta$, the operator is big in size and when the difference of $\epsilon$ and $\delta$ stays the same, the parameter $\delta$ affects the operator size. A comparison between Figures [4(b) and (c)] shows that when the difference between $\epsilon$ and $\delta$ becomes smaller and $\delta$ does not change, the operator get smaller in size. This result is consistent with the polarization equation for VTI media with weak anisotropy (Tsvankin, 2005):

$$\nu_P = \theta + B \left[ \delta + 2(\epsilon - \delta) \sin^2 \theta \right] \sin 2\theta,$$

where

$$B \equiv \frac{1}{2f} = \frac{1}{2 \left( 1 - \frac{V_{S0}^2}{V_{P0}^2} \right)}.$$

$V_{P0}$ and $V_{S0}$ are vertical P and S wave velocities, $\theta$ is the phase angle, and $\nu_P$ is the P wave polarization angle. This equation demonstrates the deviation of anisotropic polarization vectors with the isotropic ones: difference of $\epsilon$ and $\delta$ (which is approximately $\eta$ for weak anisotropy) and the parameter $\delta$ control the deviation of $\nu_P$ from $\theta$ and therefore the size of the anisotropic derivative operators.

**Operator truncation**

The derivative operators for isotropic and anisotropic media are very different in both shape and size, and the operators vary with the strength of anisotropy. In theory, analytic isotropic derivatives are point operators in the continuous limit. If one can do perfect Fourier transform to $ik_x$ and $ik_z$ (without doing the approximations to different orders of accuracy as one does in Figure 2), one gets point derivative operators. This is because $ik_x$ is constant in the $z$ direction (see Figure 6(a)), whose Fourier transform is delta function; the exact expression of $ik_x$ in the $k$ domain also makes the operator point in the $x$ direction. This makes the isotropic derivative operators point operators in the $x$ and $z$ direction. And when one applies approximations to the operators, they are compact in the space domain.

However, even if one does perfect Fourier transformation to $iU_x$ and $iU_z$ (without doing the approximations for different orders of accuracy) for VTI media, the operators will not be point operators because $iU_x$ and $iU_z$ are not constants in $z$ and $x$ directions, respectively (see Figure 6(b)). The $x$ domain operators spread out in all directions (Figures 3(b) (d) (f) and (h)).

This effect is illustrated by Figure 3. When the order of accuracy decreases, the isotropic operators become more compact (shorter in space), while the anisotropic operators do not get more compact. No matter how one improves the compactness of isotropic operators, one does not get compact anisotropic operators in the space domain by the same means.

Because the size of the anisotropic derivative operators is usually large, it is natural that one would truncate the operators to save computation. Figure 7 shows a snapshot of an elastic wavefield and corresponding derivative operators for a VTI medium with
Figure 4: 8th order anisotropic pseudo derivative operators for three VTI media: a) $\epsilon=0.25$, $\delta=-0.29$, b) $\epsilon=0.54$, $\delta=0$, and c) $\epsilon=0.2$, $\delta=0$. 

geo2009VTIModeSeparation/aniopsizemap0-order8,map1-order8,map2-order8
Figure 5: (a) The difference between the 8th and 2nd order operators (Figures 3(h) and (b)) for a VTI medium with anisotropy $\epsilon=0.25$, $\delta=-0.29$ in the $z$ and $x$ directions. (b) The difference between the 8th order anisotropic operators for a VTI medium with anisotropy $\epsilon=0.54$, $\delta=0$ (Figure 4(a)) and a VTI medium with anisotropy $\epsilon=0.25$, $\delta=-0.29$ (Figure 4(b)).
Figure 6: (a) Isotropic and (b) VTI ($\epsilon = 0.25$, $\delta = -0.29$) polarization vectors (Figure ??) projected on to the $x$ (left column) and $z$ directions (right column). The isotropic polarization vectors components in the $z$ and $x$ directions depend only on $k_z$ and $k_x$, respectively. In contrast, the anisotropic polarization vectors components are functions of both $k_x$ and $k_z$. 

[geo2009VTIModeSeparation/Matlab IsoU,AniU]
$\epsilon = 0.25$ and $\delta = -0.29$. Figure 8 shows the attempt of separation using truncated operator size of (a) $11 \times 11$, (b) $31 \times 31$ and (c) $51 \times 51$ out of the full operator size $65 \times 65$. Figure 8 shows that the truncation causes the wave-modes incompletely separated. This is because the truncation changes the directions of the polarization vectors, thus projecting the wavefield displacements onto wrong directions. Figure 9 presents the P-wave polarization vectors before and after the truncation. For a truncated operator size of $11 \times 11$, the polarization vectors deviate from the correct ones to a maximum of $10^\circ$, but even this difference makes the separation incomplete.

Figure 7: (a) A snapshot of an elastic wavefield showing the vertical (left) and horizontal (right) components for a VTI medium ($\epsilon = 0.25$ and $\delta = -0.29$). (b) 8th order anisotropic pseudo derivative operators in $z$ (left) and $x$ (right) direction for this VTI medium. The boxes show the truncation of the operator to sizes of $11 \times 11$, $31 \times 31$, and $51 \times 51$. geo2009VTIModeSeparation/separate2 uA,mop5
Figure 8: separation by 8\textsuperscript{th} order anisotropic pseudo derivative operators of different sizes: (a) $11 \times 11$, (b) $31 \times 31$, (c) $51 \times 51$, shown in Figure 7(b). The plot shows the larger the size of the operators, the better the separation is.
Figure 9: The deviation of polarization vectors by truncating the size of the space-domain operator to (a) $11 \times 11$, (b) $31 \times 31$, (c) $51 \times 51$ out of $65 \times 65$. The left column shows polarization vectors from $-1$ to $+1$ cycles in both $x$ and $z$ directions, and the right column zooms to $0.3$ to $0.7$ cycles. The green vectors are the exact polarization vectors, and the red ones are the effective polarization vectors after truncation of the operator in the $x$ domain.
EXAMPLES

I illustrate the anisotropic wave mode separation with a simple synthetic example and a more challenging elastic Sigsbee 2A model \cite{Paffenholz2002}.

Simple model

I consider a 2D isotropic model characterized by the $V_P$, $V_S$ and density shown in Figures 10(a)–(c). The model contains negative P and S velocity anomalies that triplicate the wavefields. The source is located at the center of the model. Figure 11(a) shows the vertical and horizontal components of one snapshot of the simulated elastic wavefield (generated using the 8th order finite difference solution of the elastic wave equation), Figure 11(b) shows the separation to P and S modes using $\nabla \cdot$ and $\nabla \times$ operators, and Figure 11(c) shows the mode separation obtained using the pseudo operators which are dependent on the medium parameters. A comparison of Figures 11(b) and (c) indicates that the $\nabla \cdot$ and $\nabla \times$ operators and the pseudo operators work identically well for this isotropic medium.

I then consider a 2D anisotropic model similar to the previous model shown in Figures 10(a)–(c) (with $V_P$, $V_S$ representing the vertical P and S wave velocities), and additionally characterized by the parameters $\epsilon$ and $\delta$ shown in Figures 10(d) and (e), respectively. The parameters $\epsilon$ and $\delta$ vary gradually from top to bottom and left to right, respectively. The upper left part of the medium is isotropic and the lower right part is highly anisotropic. Since the difference of $\epsilon$ and $\delta$ is great at the bottom part of the model, the $qS$ waves in this region are severely triplicated due to this strong anisotropy.

Figure 12 illustrates the pseudo derivative operators obtained at different locations in the model defined by the intersections of $x$ coordinates 0.3, 0.6, 0.9 km and $z$ coordinates 0.3, 0.6, 0.9 km. Since the operators correspond to different combination of the parameters $\epsilon$ and $\delta$, they have different forms. The isotropic operator at coordinates $x = 0.3$ km and $z = 0.3$ km, shown in Figure 12(a), is purely vertical and horizontal, while the anisotropic operators (Figure 12(b) to (i)) have “tails” radiating from the center. The operators become larger at locations where the medium is more anisotropic, for example, at coordinates $x = 0.9$ km and $z = 0.9$ km.

Figure 13(a) shows the vertical and horizontal components of one snapshot of the simulated elastic anisotropic wavefield, Figure 13(b) shows the separation to $qP$ and $qS$ modes using conventional isotropic $\nabla \cdot$ and $\nabla \times$ operators, and Figure 13(c) shows the mode separation obtained using the pseudo operators constructed using the local medium parameters. A comparison of Figure 13(b) and 13(c) indicates that the spatially-varying derivative operators successfully separate the elastic wavefields into $qP$ and $qS$ modes, while the $\nabla \cdot$ and $\nabla \times$ operators only work in the isotropic region of the model.
Figure 10: A 1.2 km $\times$ 1.2 km model with parameters (a) $V_p^0 = 3$ km/s except for a low velocity Gaussian anomaly around $x = 0.65$ km and $z = 0.65$ km, (b) $V_S^0 = 1.5$ km/s except for a low velocity Gaussian anomaly around $x = 0.65$ km and $z = 0.65$ km, (c) $\rho = 1.0$ g/cm$^3$ in the top layer and 2.0 g/cm$^3$ in the bottom layer, (d) $\epsilon$ smoothly varying from 0 to 0.25 from top to bottom, (e) $\delta$ smoothly varying from 0 to $-0.29$ from left to right. A vertical point force source is located at $x = 0.6$ km and $z = 0.6$ km shown by the dot in panels (b), (c), (d), and (e). The dots in panel (a) correspond to the locations of the anisotropic operators shown in Figure 12.
Figure 11: (a) One snapshot of the isotropic wavefield modeled with a vertical point force source at \(x=0.6\) km and \(z=0.6\) km for the model shown in Figure 10, (b) isotropic P and S wave modes separated using \(\nabla \cdot\) and \(\nabla \times\), and (c) isotropic P and S wave modes separated using pseudo derivative operators. Both (b) and (c) show good separation results.
Figure 12: The 8th order anisotropic pseudo derivative operators in the z and x directions at the intersections of x=0.3, 0.6, 0.9 km and z=0.3, 0.6, 0.9 km for the model shown in Figure [10].

Sigsbee model

My second model (Figure [14]) uses an elastic anisotropic version of the Sigsbee 2A model [Paffenholz et al., 2002]. In the modified model, $V_{P0}$ is taken from the original model, the $V_{P0}/V_{S0}$ ratio ranges from 1.5 to 2, the parameter $\epsilon$ ranges from 0 to 0.48 (Figure [14(d)]) and the parameter $\delta$ ranges 0 from to 0.10 (Figure [14(e)]). The model is isotropic in the salt and the top part of the model. A vertical point force source is located at coordinates $x = 14.5$ km and $z = 5.3$ km to simulate the elastic anisotropic wavefield.

Figure ?? shows one snapshot of the modeled elastic anisotropic wavefields using the model shown in Figure [14]. Figure ?? illustrates the separation of the anisotropic elastic wavefields using the $\nabla \cdot$ and $\nabla \times$ operators, and Figure ?? illustrates the separation using my pseudo derivative operators. Figure ?? shows the residual of unseparated P and S wave modes, such as at coordinates $x = 13$ km and $z = 7$ km in the $qP$ panel and at $x = 11$ km and $z = 7$ km in the $qS$ panel. The residual of S waves in the $qP$ panel of Figure ?? is very significant because of strong reflections from the salt bottom. This extensive residual can be harmful to under-salt elastic or even acoustic migration, if not removed completely. In contrast, Figure ?? shows the $qP$ and $qS$ modes better separated, demonstrating the effectiveness of the anisotropic pseudo derivative operators constructed using the local medium parameters. These wavefields composed of well separated $qP$ and $qS$ modes are essential to producing
Figure 13: (a) One snapshot of the anisotropic wavefield modeled with a vertical point force source at $x=0.6$ km and $z=0.6$ km for the model shown in Figure 10 (b) anisotropic $qP$ and $qS$ modes separated using $\nabla \cdot$ and $\nabla \times$, and (c) anisotropic $qP$ and $qS$ modes separated using pseudo derivative operators. The separation of wavefields into $qP$ and $qS$ modes in (b) is not complete, which is obvious at places such as at coordinates $x = 0.4$ km $z = 0.9$ km. In contrast, the separation in (c) is much better, because the correct anisotropic derivative operators are used.
clean seismic images.

In order to test the separation with a homogeneous assumption of anisotropy in the model, I show in Figure ?? the separation with $\epsilon = 0.3$ and $\delta = 0.1$ in the $k$ domain. This separation assumes a model with homogeneous anisotropy. The separation shows that there is still residual in the separated panels. Although the residual is much weaker compared to separating using an isotropic model, it is still visible at locations such as at coordinates $x = 13$ km and $z = 7$ km, and $x = 13$ km and $z = 4$ km in the $qP$ panel and at $x = 16$ km and $z = 2.5$ km in the $qS$ panel.

![Figure 14: A Sigsbee 2A model in which (a) is the P wave velocity (taken from the original Sigsbee 2A model (Paffenholz et al., 2002)), (b) is the S wave velocity, where $V_{P0}/V_{S0}$ ratio ranges from 1.5 to 2.0, (c) is the density ranging from 1.0 g/cm$^3$ to 2.2 g/cm$^3$, (d) is the parameter $\epsilon$ ranging from 0.20 to 0.48, and (e) is the parameter $\delta$ ranging from 0 to 0.10 in the rest of the model.](geo2009VTIModeSeparation/sigsbee vp,vs,ro,epsilon,delta)
Figure 15: Anisotropic wavefield modeled with a vertical point force source at $x = 14.3$ km and $z = 5.3$ km for the model shown in Figure 14.

Figure 16: Anisotropic $qP$ and $qS$ modes separated using $\nabla \cdot$ and $\nabla \times$ for the vertical and horizontal components of the elastic wavefields shown in Figure ???. Residuals are obvious at places such as at coordinates $x = 13$ km and $z = 7$ km in the $qP$ panel and at $x = 11$ km and $z = 7$ km in the $qS$ panel.
Figure 17: Anisotropic $q_P$ and $q_S$ modes separated using pseudo derivative operators for the vertical and horizontal components of the elastic wavefields shown in Figure ???. They show better separation of $q_P$ and $q_S$ modes.

Figure 18: Anisotropic $q_P$ and $q_S$ modes separated in the $k$ domain for the vertical and horizontal components of the elastic wavefields shown in Figure ???. The separation assumes $\epsilon = 0.3$ and $\delta = 0.1$ throughout the model. The separation is incomplete. Residuals are still visible at places such at coordinates $x = 13$ km and $z = 7$ km, and $x = 13$ km and $z = 4$ km in the $q_P$ panel and at $x = 16$ km and $z = 2.5$ km in the $q_S$ panel.
DISCUSSION

The separation of P and S wave-modes is based on the projection of elastic wavefields onto their respective polarization vectors. For VTI media, P and S mode polarization vectors can be conveniently obtained by solving the Christoffel equation. The Christoffel equation is a plane-wave solution to the elastic wave equation. Since the displacements, velocity and acceleration field have the same form of elastic wave equation, the separation algorithm applies to all these wavefields. The P and SV mode separation can be extended to TTI (transverse isotropy with a tilted symmetry axis) media by solving a TTI Christoffel matrix, and obtain TTI separators. Physically, the TTI media is just a rotation of VTI media.

In TI media, SV and SH waves are uncoupled most of the time, where SH wave is polarized out of plane. One only needs to decompose P and SV modes in the vertical plane. The plane wave solution is sufficient for most TI media, except for a special case where there exists a singularity point at an oblique propagation angle in the vertical plane (a line singularity in 3D), at which angle SV and SH wave velocities coincide. At this point, the SV wave polarization is not uniquely defined by Christoffel equation. S waves at the singularity are polarized in a plane orthogonal to the P wave polarization vector. However, this is not a problem since we define SV waves polarized in vertical planes only, therefore I remove the singularity by using the cylindrical coordinates. This situation is similar to S wave-mode coupling in orthorhombic media, where there is at least one singularity in a quadrant. However, as pointed out by Dellinger and Etgen (1990), the singularity in orthorhombic media is a global property of the media and cannot be removed, therefore the separation using polarization vectors in 3D orthorhombic media is not straightforward.

The anisotropic derivative operators depend on the anisotropic medium parameters. In Figure 19, I show how sensitive the separation is to the medium parameters. One elastic wavefield snapshot is shown in Figure 7(a) for a VTI medium with \( \frac{V_{P0}}{V_{S0}} = 2 \) and \( \epsilon = 0.25 \), \( \delta = -0.29 \). I try to separate the P and SV modes with (a) \( \epsilon = 0.4 \), \( \delta = -0.1 \), (b) \( \epsilon = 0 \), \( \delta = -0.3 \) and (c) \( \epsilon = 0 \), \( \delta = 0 \). The separation shows that parameters (a) have good separation, showing the difference in \( \epsilon \) and \( \delta \) is important. The worst case scenario is shown by parameters (c), where isotropy is assumed for this VTI medium.

CONCLUSIONS

I present a method of obtaining spatially-varying pseudo derivative operators with application to wave mode separation in anisotropic media. The main idea is to utilize polarization vectors constructed in the wavenumber domain using the local medium parameters and then transform these vectors back to the space domain. The main advantage of applying the pseudo derivative operators in the space domain constructed in this way is that they are suitable for heterogeneous media. The wave mode separators obtained using this method are spatially-variable filtering operators and can
Figure 19: P and SV wave mode separation for a snapshot shown in Figure 7(a). The true medium parameters are $\epsilon = 0.25$, $\delta = -0.29$. The separation assumes medium parameters of (a) $\epsilon = 0.4$, $\delta = -0.1$, (b) $\epsilon = 0$, $\delta = -0.3$, and (c) $\epsilon = 0$, $\delta = 0$. Hard clipping was applied to show the weak events. The plot shows that different estimate of anisotropy parameters has influence on the wave mode separation.
be used to separate wavefields in VTI media with an arbitrary degree of anisotropy. This methodology is applicable for elastic RTM in heterogeneous anisotropic media.

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Wide-azimuth angle gathers for wave-equation migration

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ABSTRACT
Extended common-image-point-gathers (CIP) contain all the necessary information for decomposition of reflectivity as a function of the reflection and azimuth angles at selected locations in the subsurface. This decomposition operates after the imaging condition applied to wavefields reconstructed by any type of wide-azimuth migration method, e.g. using downward continuation or time reversal. The reflection and azimuth angles are derived from the extended images using analytic relations between the space-lag and time-lag extensions. The transformation amounts to a linear Radon transform applied to the CIPs obtained after the application of the extended imaging condition. If information about the reflector dip is available at the CIP locations, then only two components of the space-lag vectors are required, thus reducing computational cost and increasing the affordability of the method. Applications of this method include the study of subsurface illumination in areas of complex geology where ray-based methods are not usable, and the study of amplitude variation with reflection and azimuth angles if the subsurface subsurface illumination is sufficiently dense. Migration velocity analysis could also be implemented in the angle domain, although an equivalent implementation in the extended domain is cheaper and more effective.

INTRODUCTION
In regions characterized by complex subsurface structure, wave-equation depth migration is a powerful tool for accurately imaging the earth’s interior. The quality of the final image greatly depends on the quality of the velocity model and on the quality of the technique used for wavefield reconstruction in the subsurface (Gray et al., 2001).

However, structural imaging is not the only objective of wave-equation imaging. It is often desirable to construct images depicting reflectivity as a function of reflection angles. Such images not only highlight the subsurface illumination patterns, but could potentially be used for image postprocessing for amplitude variation with angle.

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analysis. Furthermore, angle domain images can be used for tomographic velocity updates.

Angle gathers can be produced either using ray methods (Xu et al., 1998; Brandsberg-Dahl et al., 2003) or by using wavefield methods (de Bruin et al., 1990; Mosher et al., 1997; Prucha et al., 1999; Xie and Wu, 2002; Rickett and Sava, 2002; Sava and Fomel, 2003; Biondi and Symes, 2004; Wu and Chen, 2006). Gathers constructed with these methods have similar characteristics since they simply describe the reflectivity as a function of incidence angles at the reflector. However, as indicated by Stolk and Symes (2004), even in perfectly known but strongly refracting media angle gathers are damaged by undersampling of data on the surface, regardless of the method used for their construction. In this paper, we address the problem of wavefield-based angle decomposition.

Angle decomposition can be applied either before or after the application of an imaging condition. The two classes of methods differ by the objects used to study the angle-dependent illumination of subsurface geology. The methods operating before the imaging condition decompose the extrapolated wavefields from the source and receivers (de Bruin et al., 1990; Mosher et al., 1997; Prucha et al., 1999; Wu and Chen, 2006). This type of decomposition is costly since it operates on individual wavefields characterized by complex multipathing. In contrast, the methods operating after the imaging condition decompose the images themselves which are represented as a function of space and additional parameters, typically referred to as extensions (Rickett and Sava, 2002; Sava and Fomel, 2003, 2006; Sava and Vasconcelos, 2011). In the end, the various classes of methods lead to similar representations of the angle-dependent reflectivity represented by the so-called scattering matrix. The main differences lie in the complexity of the decomposition and in the cost required to achieve this result. In this paper, we focus on angle decomposition of extended images.

Conventionally, angle-domain imaging uses common-image-gathers (CIGs) describing the reflectivity as a function of reflection angles and a space axis, typically the depth axis. An alternative way of constructing angle-dependent reflectivity is based on common-image-point-gathers (CIP) selected at various positions in the subsurface. As pointed out by Sava and Vasconcelos (2011), CIPs are advantageous because they sample the image at the most relevant locations (along the main reflectors), they avoid computations at locations that are not useful for further analysis (inside salt bodies), they can have higher density at locations where the structure is more complex and lower density in areas of poor illumination, and they avoid the depth bias typical for gathers constructed as a function of the depth axis. In this paper, we focus on angle decomposition using extended CIPs.

A recent development in wave-equation imaging is the use of wide-azimuth data (Regone, 2006; Michell et al., 2006; Clarke et al., 2006). Imaging with such data poses additional challenges for angle-domain imaging, mainly arising from the larger data size and the interpretation difficulty of data of higher dimensionality. Several techniques have been proposed for wide-azimuth angle decomposition, including ray-based methods (Koren et al., 2008) and wavefield methods using wavefield decomposition
before imaging (Zhu and Wu, 2010; Biondi and Tisserant, 2004) or after imaging (Sava and Fomel, 2005). Here, we complete the set of techniques available for angle gather construction by describing an algorithm applicable to extended common-image-point-gathers.

**IMAGING CONDITIONS**

Conventional seismic imaging is based on the concept of single scattering. Under this assumption, waves propagate from seismic sources, interact with discontinuities and return to the surface as reflected seismic waves. We commonly speak about a “source” wavefield, originating at the seismic source and propagating in the medium prior to any interaction with discontinuities, and a “receiver” wavefield, originating at discontinuities and propagating in the medium to the receivers (Berkhout, 1982; Clærbout, 1985). The two wavefields kinematically coincide at discontinuities.

We can formulate imaging as a process involving two steps: the wavefield reconstruction and the imaging condition. The key elements in this imaging procedure are the source and receiver wavefields, $W_s$ and $W_r$, which are 4-dimensional objects as a function of space $x = \{x, y, z\}$ and time $t$, or as a function of space and frequency $\omega$. For imaging, we need to analyze if the wavefields match kinematically in time and then extract the reflectivity information using an imaging condition operating along the space and time axes.

A conventional cross-correlation imaging condition (cIC) based on the reconstructed wavefields can be formulated in the time or frequency domain as the zero lag of the cross-correlation between the source and receiver wavefields (Clærbout, 1985):

$$R(x) = \sum_{\text{shots}} \sum_t W_s(x,t) W_r(x,t) \quad (1)$$

$$= \sum_{\text{shots}} \sum_\omega \overline{W_s(x,\omega)} W_r(x,\omega), \quad (2)$$

where $R$ represents the migrated image and the over-line represents complex conjugation. This operation exploits the fact that portions of the source and receiver wavefields match kinematically at subsurface positions where discontinuities occur. Alternative imaging conditions use deconvolution of the source and receiver wavefields, but we do not elaborate further on this subject since the differences between cross-correlation and deconvolution are not central for this paper.

An extended imaging condition preserves in the output image certain acquisition (e.g. source or receiver coordinates) or illumination (e.g. reflection angle) parameters (Clayton and Stolt, 1981; Clærbout, 1985; Stolt and Weglein, 1985; Weglein and Stolt, 1999). In shot-record migration, the source and receiver wavefields are reconstructed on the same computational grid at all locations in space and all times or frequencies, therefore there is no a-priori wavefield separation that can be transferred to the output image. In this situation, the separation can be constructed by correlation of the
wavefields from symmetric locations relative to the image point, measured either in space (Rickett and Sava, 2002; Sava and Fomel, 2005) or in time (Sava and Fomel, 2006). This separation essentially represents local cross-correlation lags between the source and receiver wavefields. Thus, an extended cross-correlation imaging condition (eIC) defines the image as a function of space and cross-correlation lags in space and time. This imaging condition can also be formulated in the time and frequency domains (Sava and Vasconcelos, 2011):

\[
R(x, \lambda, \tau) = \sum_{\text{shots}} \sum_{t} W_s(x - \lambda, t - \tau) W_r(x + \lambda, t + \tau) \quad (3)
\]

\[
= \sum_{\text{shots}} \sum_{\omega} e^{2i\omega \tau} W_s(x - \lambda, \omega) W_r(x + \lambda, \omega) . \quad (4)
\]

Equations 1-2 represent a special case of equations 3-4 for \( \lambda = 0 \) and \( \tau = 0 \). Assuming that all errors accumulated in the incorrectly-reconstructed wavefields are due to the velocity model, the extended images could be used for velocity model building by exploiting semblance properties emphasized by the space-lags (Biondi and Sava, 1999; Shen et al., 2003; Sava and Biondi, 2004a,b) and focusing properties emphasized by the time-lag (Paye and Jeannot, 1986; MacKay and Abma, 1992, 1993; Nemeth, 1995, 1996). Furthermore, these extensions can be converted to reflection angles (Weglein and Stolt, 1999; Sava and Fomel, 2003, 2006), thus enabling analysis of amplitude variation with angle for images constructed in complex areas using wavefield-based imaging.

Typically, angle decomposition with extended images uses common image gathers, i.e. representations of (a subset of) the extensions as a function of a space axis, typically the depth axis. As pointed out by Sava and Vasconcelos (2011), this approach suffers from major drawbacks. Common-image-gathers are appropriate for nearly horizontal structures and they are computationally wasteful since they require unnecessary calculations, e.g. inside massive salt bodies. In contrast, the common-image-point-gathers advocated by Sava and Vasconcelos (2011) are constructed at selected points in the image, thus eliminating unnecessary calculations, and they can accommodate arbitrary orientations of the reflectors.

In this paper, we use extended common-image-point-gathers to extract angle-dependent reflectivity at individual points in the image. The method described in the following section is appropriate for 3D wide-azimuth wave-equation imaging. The problem we are solving is to decompose extended CIPs as a function of azimuth \( \phi \) and reflection \( \theta \) angles at selected points in the image. In general, the input for such decomposition are gathers in the \( \{\lambda, \tau\} \) domain, and the output are gathers in the \( \{\phi, \theta, \tau\} \) domain:

\[
R(\lambda, \tau) \rightarrow R(\phi, \theta, \tau) . \quad (5)
\]

Here we address a special case of this decomposition which is appropriate for imaging with correct velocity. In this case, all the energy in the output CIPs concentrates at \( \tau = 0 \), so we can focus our attention on a particular case of the decomposition which does not preserve the time-lag variable in the output: \( R(\lambda, \tau) \rightarrow R(\phi, \theta) \). We
omit the dependence of the extended images on space \( (x) \) to highlight the fact that
the decomposition can be performed independently at various points in the image.
The topic of angle decomposition when the gathers are constructed with incorrect
velocity remains outside the scope of this paper. However, we note that the angle
decomposition in such situation is not necessary, since semblance optimization can be
implemented based on the image extensions directly [Shen and Symes, 2008; Symes,
2009].

In the remainder of the paper, we show that all the necessary information for this
decomposition is available after wave-equation migration, regardless of its implement-
tation, e.g by depth or time extrapolation. A pre-requisite for this decomposition is
the moveout function characterizing individual shots, as discussed in the following
section. We then show how the moveout information can be used for angle decompo-
sition and illustrate the method with simple and complex synthetic examples.

**MOVEOUT FUNCTION**

In this section, we derive the formula for the moveout function characterizing re-
fections in the extended \( \{\lambda, \tau\} \) domain. The purpose of this derivation is to find a
procedure for angle decomposition, i.e. a representation of reflectivity as a function
of reflection and azimuth angles.

An implicit assumption made by all methods of angle decomposition is that we
can describe the reflection process by locally planar objects. Such methods assume
that (locally) the reflector is a plane, and that the incident and reflected wavefields are
also (locally) planar. Only with these assumptions we can define vectors in-between
which we measure angles like the angles of incidence and reflection, as well as the
azimuth angle of the reflection plane. Our method uses this assumption explicitly.
However, we do not assume that the wavefronts are planar. Instead, we consider each
(complex) wavefront as a superposition of planes with different orientations. In the
following, we discuss how each one of these planes would behave during the extended
imaging and angle decomposition. Thus, our method applies equally well for simple
and complex wavefields characterized by multipathing.

We define the following unit vectors to describe the reflection geometry and the
conventional and extended imaging conditions:

- \( \mathbf{n} \): a unit vector aligned with the reflector normal;
- \( \mathbf{a} \): a unit vector representing the projection of the azimuth vector \( \mathbf{v} \) in the
  reflector plane;
- \( \mathbf{n}_s \): a unit vector orthogonal to the source wavefront;
- \( \mathbf{n}_r \): a unit vector orthogonal to the receiver wavefront;
• $\mathbf{q}$: a unit vector at the intersection of the reflection plane and the reflector plane.

By construction, vectors $\mathbf{n}$, $\mathbf{n}_s$, $\mathbf{n}_r$ and $\mathbf{q}$ are co-planar and vectors $\mathbf{n}$ and $\mathbf{q}$ are orthogonal, Figure 1.

With these definitions, the (planar) source and receiver wavefields are given by the expressions:

$$\mathbf{n}_s \cdot \mathbf{x} = v t_s , \quad \text{(6)}$$
$$\mathbf{n}_r \cdot \mathbf{x} = v t_r . \quad \text{(7)}$$

Here, $\mathbf{x}$ are space coordinates, $t_s$ and $t_r$ are times defining the planes under consideration, and $v$ represents velocity. Equations 6 and 7 define the conventional imaging condition given by equations 1 and 2. This condition states that an image is formed when the source and receiver wavefields are time-coincident at reflection points. In Equations 6 and 7 we explicitly impose the condition that the source and receiver planes and the reflector plane intersect at the image point.

Similarly, we can rewrite the extended imaging condition using the planar approximation of the source and receiver wavefields using the expressions:

$$\mathbf{n}_s \cdot (\mathbf{x} - \lambda) = v (t_s - \tau) , \quad \text{(8)}$$
$$\mathbf{n}_r \cdot (\mathbf{x} + \lambda) = v (t_r + \tau) . \quad \text{(9)}$$

As discussed earlier, $\lambda$ and $\tau$ are space- and time-lags, and $v$ represents the local velocity at the image point, assumed to be constant in the immediate vicinity of this point. This assumption is justified by the need to operate with planar objects, as indicated earlier. With this construction, the source and receiver planes are shifted relative to one-another by equal quantities in the positive and negative directions and in space and time, equations 3-4.

We can eliminate the space variable $\mathbf{x}$ by substituting equation 6 in equation 8 and equation 7 in equation 9:

$$\mathbf{n}_s \cdot \lambda = v \tau , \quad \text{(10)}$$
$$\mathbf{n}_r \cdot \lambda = v \tau . \quad \text{(11)}$$

Furthermore, we can re-arrange the system given by equations 10 and 11 by sum and difference of the equations:

$$\mathbf{n}_s + \mathbf{n}_r \cdot \lambda = 2 v \tau , \quad \text{(12)}$$
$$\mathbf{n}_s - \mathbf{n}_r \cdot \lambda = 0 . \quad \text{(13)}$$

So far, we have not assumed any relation between the vectors characterizing the source and receiver planes, $\mathbf{n}_s$ and $\mathbf{n}_r$. However, if the source and receiver wavefields
correspond to a reflection from a planar interface, these vectors are not independent of one-another, but are related by Snell’s law which can be formulated as

\[ n_r = n_s - 2(n_s \cdot n)n. \]  

(14)

This relations follows from geometrical considerations and it is based on the conservation of ray vector projection along the reflector. Equation (14) is only valid for PP reflections in an isotropic medium.

Substituting Snell’s law into the system (12-13) and after trivial manipulations of the equations, we obtain the system:

\[ [n_s - (n_s \cdot n)n] \cdot \lambda = v\tau, \]  

(15)

\[(n_s \cdot n)(n \cdot \lambda) = 0.\]  

(16)

In general, the plane characterizing the source wavefield is not orthogonal to the reflection plane (there would be no reflection in that case), therefore we can simplify equation (16) by dropping the term \((n_s \cdot n) \neq 0\). Moreover, we can replace in equation (15) the expression in the square bracket with the quantity \(q \sin \theta\), where \(q\) is the unit vector characterizing the line at the intersection of the reflection and reflector planes, and \(\theta\) is the reflection angle contained in the reflection plane. With these simplifications, the system (15-16) can be re-written as:

\[(q \cdot \lambda) \sin \theta = v\tau, \]  

(17)

\[n \cdot \lambda = 0.\]  

(18)

The system (17-18) allows for a straightforward physical interpretation of the extended imaging condition. First, the expression (18) indicates that of all possible space-lags that can be applied to the reconstructed wavefields, the only ones that contribute to the extended image are those for which the space-lag vector \(\lambda\) is orthogonal to the reflector normal vector \(n\). Furthermore, assuming that the space-shift applied to the source and receiver planes is contained in the reflector plane, i.e. \(\lambda \perp n\), then the expression (17) describe the moveout function in an extended gather as a function of the space-lag \(\lambda\), the time lag \(\tau\), the reflection angle \(\theta\), the orientation vector \(q\). The vector \(q\) is orthogonal to the reflector normal and depends on the reflection azimuth angle \(\phi\).

Figures 1-3 illustrate the process involved in the extended imaging condition and describe pictorially its physical meaning. Figure 1 shows the source and receiver planes, as well as the reflector plane together with their unit vector normals. Figure 2 shows the source and receiver planes displaced by the space lag vector \(\lambda\) contained in the reflector plane, as indicated by equation (18). The displaced planes do not intersect at the reflection plane, thus they do not contribute to the extended image at this point. However, with the application of time shifts with the quantity \(\tau = (q \cdot \lambda) \sin \theta/v\), i.e. a translation in the direction of plane normals, the source and receiver planes are restored to the image point, thus contributing to the extended image, Figure 3.
Figure 1: The reflector plane (of normal $n$), together with the source and receiver planes (of normals $n_s$ and $n_r$, respectively). The figure represents the source/receiver planes in their original position, i.e. as obtained by wavefield reconstruction.
Figure 2: The reflector plane (of normal $\mathbf{n}$), together with the source and receiver planes (of normals $\mathbf{n}_s$ and $\mathbf{n}_r$, respectively). The figure represents the source/receiver planes displaced with the space-lag $\lambda$ constrained in the reflector plane.
Figure 3: The reflector plane (of normal $\mathbf{n}$), together with the source and receiver planes (of normals $\mathbf{n}_s$ and $\mathbf{n}_r$, respectively). The figure represents the source/receiver planes displaced with space-lag $\lambda$ and time-lag $\tau$. The space and time-lags are related by equation 17.
ANGLE DECOMPOSITION

In this section we discuss the steps required to transform lag-domain CIPs into angle-domain CIPs using the moveout function derived in the preceding section. We also present the algorithm used for angle decomposition and illustrate it using a simple 3D model of a horizontal reflector in a medium with constant velocity which allows us to validate analytically the procedure.

The outer loop of the algorithm is over the CIPs evaluated during migration. The angle decompositions of individual CIPs are independent of one-another, therefore the algorithm is easily parallelizable over the outer loop. At every CIP, we need to access the information about the reflector normal \((\mathbf{n})\) and about the local velocity \((v)\). The reflector dip information can be extracted from the conventional image, and the velocity is the same as the one used for migration.

Prior to the angle decomposition, we also need to define a direction relative to which we measure the reflection azimuth. This direction is arbitrary and depends on the application of the angle decomposition. Typically, the azimuth is defined relative to a reference direction (e.g. North). Here, we define this azimuth direction using an arbitrary vector \(\mathbf{v}\). Using the reflector normal \((\mathbf{n})\) we can build the projection of the azimuth vector \((\mathbf{a})\) in the reflector plane as

\[
\mathbf{a} = (\mathbf{n} \times \mathbf{v}) \times \mathbf{n}.
\]

This construction assures that vector \(\mathbf{a}\) is contained in the reflector plane (i.e. it is orthogonal on \(\mathbf{n}\)) and that it is co-planar with vectors \(\mathbf{n}\) and \(\mathbf{v}\), Figure 1. Of course, this construction is just one of the many possible definitions of the azimuth reference. In the following, we measure the azimuth angle \(\phi\) relative to vector \(\mathbf{a}\) and the reflection angle \(\theta\) relative to the normal to the reflector given by vector \(\mathbf{n}\).

Then, for every azimuth angle \(\phi\), using the reflector normal \((\mathbf{n})\) and the azimuth reference \((\mathbf{a})\), we can construct the trial vector \(\mathbf{q}\) which lies at the intersection of the reflector and the reflection planes. We scan over all possible vectors \(\mathbf{q}\), although only one azimuth corresponds to the reflection from a given shot. This scan ensures that we capture the reflection information from all shots in the survey. Given the reflector normal (the axis of rotation) and the trial azimuth angle \(\phi\), we can construct the different vectors \(\mathbf{q}\) by the application of the rotation matrix

\[
Q(\mathbf{n}, \phi) = \begin{bmatrix}
\frac{n_x^2 + (n_y^2 + n_z^2) \cos \phi}{n_y n_x (1 - \cos \phi) + n_z \sin \phi} & n_x n_y (1 - \cos \phi) - n_z \sin \phi & n_x n_z (1 - \cos \phi) + n_y \sin \phi \\
n_y n_x (1 - \cos \phi) + n_z \sin \phi & \frac{n_y^2 + (n_z^2 + n_x^2) \cos \phi}{n_z n_x (1 - \cos \phi) - n_y \sin \phi} & n_y n_z (1 - \cos \phi) - n_x \sin \phi \\
n_z n_x (1 - \cos \phi) - n_y \sin \phi & n_z n_y (1 - \cos \phi) + n_x \sin \phi & \frac{n_z^2 + (n_x^2 + n_y^2) \cos \phi}{n_x^2 + (n_y^2 + n_z^2) \cos \phi}
\end{bmatrix}
\]

(20)

to the azimuth reference vector \(\mathbf{a}\), i.e.

\[
\mathbf{q} = Q(\mathbf{n}, \phi) \mathbf{a}.
\]

(21)

In this formulation, the normal vector \(\mathbf{n}\) of components \(\{n_x, n_y, n_z\}\) can take arbitrary orientations and does not need to be normalized. Then, for every reflection
angle $\theta$, we map the lag-domain CIP to the angle-domain by summation over the surface defined by equation 17. This operation represents a planar Radon transform (a slant-stack) over an analytically-defined surface in the $\{\lambda, \tau\}$ space. The output is the representation of the CIP in the angle-domain. In order to preserve the signal bandwidth, the slant-stack needs to use a “rho filter” which compensates the high frequency decay caused by the summation (Claerbout 1976). The explicit algorithm for angle decomposition is given in Appendix A.

Consider a simple 3D model consisting of a horizontal reflector in a constant velocity medium. We simulate one shot in the center of the model at coordinates $x = 4$ km and $y = 4$ km, with receivers distributed uniformly on the surface on a grid spaced at every 20 m in the $x$ and $y$ directions. We use time-domain finite-differences for modeling. Figure 4 represents the image obtained by wave-equation migration of the simulated shot using downward continuation. The illumination is limited to a narrow region around the shot due to the limited array aperture.

Figure 4: The image obtained for a horizontal reflector in constant velocity using one shot located in the center of the model.

Figures 5(a)-5(d) depict CIPs obtained by migration of the simulated shot at the reflector depth and at coordinates $\{x, y\}$ equal to {3.2, 3.2} km, {3.2, 4.8} km, {4.8, 4.8} km and {4.8, 3.2} km, respectively. For these CIPs, the reflection angle is invariant $\theta = 48.5^\circ$, but the azimuth angles relative to the $x$ axis are $-135^\circ$, $+135^\circ$, $+45^\circ$ and $-45^\circ$, respectively. Figures 5(e)-5(h) show the angle decomposition in polar coordinates. Here, we use the trigonometric convention to represent the azimuth angle $\phi$ and we represent the reflection angle in every azimuth in the radial direction (with normal incidence at the center of the plot). Each radial line corresponds to 30$^\circ$ and each circular contour corresponds to 15$^\circ$.

Similarly, Figures 6(a)-6(d) depict CIPs obtained by migration of the simulated shot the reflector depth and at coordinates $\{x, y\}$ equal to {2.8, 2.8} km, {3.2, 3.2} km, {3.6, 3.6} km and {4.0, 4.0} km, respectively. For these CIPs, the azimuth angle is invariant $\phi = -135^\circ$, but the reflection angles relative to the reflector normal are $59.5^\circ$, $48.5^\circ$, $29.5^\circ$, and $0^\circ$ respectively.

In all examples, the decomposition angles correspond to the theoretical values, thus confirming the validity of our decomposition.
Figure 5: Illustration of CIP angle decomposition for illumination at fixed reflection angle. Panels (a)-(d) show lag-domain CIPs, and panels (e)-(f) show angle-domain CIPs in polar coordinates. The angles $\phi$ and $\theta$ are indexed along the contours using the trigonometric convention and along the radial lines increasing from the center.
Figure 6: Illustration of CIP angle decomposition for illumination at fixed azimuth angle. Panels (a)-(d) show lag-domain CIPs, and panels (e)-(h) show angle-domain CIPs in polar coordinates. The angles $\phi$ and $\theta$ are indexed along the contours using the trigonometric convention and along the radial lines increasing from the center.
EXAMPLES

We illustrate the method discussed in the preceding section with common-image-point-gathers constructed using the wide-azimuth SEAM data. Figure 7 shows the velocity model in the area used for imaging. For demonstration, we consider 16 shots located at the locations of the thick dots in Figure 9(d). The thin dots represent all the 357 shots available in one of the SEAM data subsets. The solid lines in Figures 9(a)–9(b) depict the decimated receiver lines for each of the 3 shots shown. In all panels 9(a)–9(d), the large dot indicates the surface projection of the CIP used for illustration, located at coordinates \( x, y, z = \{23.450, 11.425, 2.38\} \) km. For this example we consider the azimuth reference vector oriented in the \( x \) direction, i.e. \( \mathbf{v} = \{1, 0, 0\} \).

![Figure 7: A subset of the SEAM velocity model used for the imaging example in Figures 8–11(d).](geo2011WideAzimuthAngleDecomposition/seam357 velo)

![Figure 8: Conventional image obtained using wavefield extrapolation with the 16 shots shown in Figure 9(d).](geo2011WideAzimuthAngleDecomposition/seam357 cstk)
Figure 9: Geometry of SEAM imaging experiment. Panels (a)-(c) show the position of one shot and the associated receiver lines (decimated by a factor of 30 in the $y$ direction. Panel (d) shows the locations of the 16 shots used for creating the image shown in Figure 8.
Figures 10(a)-10(c) show the extended image obtained at the CIP location indicated earlier using migration by downward continuation. The extended image cubes use 41 grid points in the $h_x$ and $h_y$ directions sampled on the image grid, i.e. at every 30 m, and 31 grid points in the $\tau$ direction sampled on the data grid, i.e. at every 8 ms. The vertical lag $h_z$ is not computed in this example, since the analyzed reflector is nearly-horizontal. This lag is computed in the decomposition process from the horizontal lag and from the known information about the normal to the reflector at the given position. Figure 10(d) shows the extended image obtained for all 16 shots used for imaging. Although here we show the extended image cubes for independent shots, in practice these cubes need not be computed separately – the decomposition separates the information corresponds to different angles of incidence, as shown in this simple example.

Finally, Figures 11(a)-11(d) show the angle-domain decomposition of the extended image cubes shown in Figures 10(a)-10(d), respectively. In these plots, the circles indicating the reflection angles are drawn at every 5° and the radial lines indicating the azimuth directions are drawn at every 15°. Given the sparse shot sampling, the CIP is sparsely illuminated, but at the correct reflection and azimuth angles.

**DISCUSSION**

We do not suggest in this paper that the wavefields used for imaging are planar prior to the interaction with the reflector. In complex geology, such an assumption would be unrealistic. However, a wavefield of arbitrary shape can be thought of as a superposition of plane waves propagating in various directions, either because the wavefronts characterizing the wavefields have curvature, or because the wavefields have triplicated during propagation. Each incident plane has a corresponding reflected plane related through Snell’s law. Some angle decomposition techniques make use explicitly of a planar decomposition of the wavefields, followed of selection through thresholding of the most energetic plane (Xu et al., 2010). In contrast, we rely on the fact that all planar components of the wavefields have been transformed as planar events in the extended images and rely on slant-stacks or equivalent methods to separate them as a function of azimuth and reflection angles.

As indicated in the preceding sections, we do not need to compute all space-lags at the considered CIP positions. We could compute just two of them, e.g. $\lambda_x$ and $\lambda_y$ as shown in the examples of this paper, and then reconstruct the third lag using the information given by the reflector normal at the CIP position, equation 18. If the reflector is nearly vertical, it may be more relevant to compute the vertical and one horizontal space-lags. Alternatively, we could avoid computing the reflector normal vector from the conventional image, but instead compute all three components of the space-lag vector $\hat{\lambda}$. In this case, as indicated by Sava and Vasconcelos (2011), we could estimate the reflector dip from the lag information prior to the angle decomposition.

We have also noted earlier in the paper that the relevant space-lags are constructed...
Figure 10: Extended image cubes for the SEAM imaging experiment. Panels (a)-(c) show extended image cubes at the same location for 3 different shots, and panel (d) shows the extended image obtained for all 16 shots considered in this experiment.
Figure 11: Reflectivity as a function of reflection and azimuth angles for the SEAM imaging experiment. Panels (a)-(c) show the angle-domain CIPs at the same location for 3 different shots, and panel (d) shows the angle-domain CIP obtained for all 16 shots considered in this experiment. The angles $\phi$ and $\theta$ are indexed along the contours using the trigonometric convention and along the radial lines increasing from the center.

geo2011WideAzimuthAngleDecomposition/seam357 cang-037449,cang-041729,cang-043873,cang
in the reflector plane. This fact is a direct consequence of the fact that we have considered equal but with opposite sign time-shift of the source and receiver wavefields. Without this convention, the angle decomposition problem becomes more complex. In our experience to date, we did not find the need to relax this requirement.

The angle-domain CIPs accurately indicate the sampling of the reflector as a function of azimuth and reflection angles. If the shot distribution is sparse, or if the sub-surface geology creates shadow zones, the illumination is also sparse. This is both beneficial, assuming that the angle-domain CIPs are used to evaluate illumination, but it can also be a drawback if the angle-domain CIPs are used for AVA or MVA. However, a sparse sampling of a reflector is not a feature of the angle decomposition, but a feature of the acquisition geometry. Neither our, nor any other angle decomposition, can compensate for the lack of adequate data illuminating the subsurface on a dense angular grid.

Finally, we note that the most likely applications for angle decomposition in complex geology is the study of the reflector illumination itself. Assuming that the sampling is sufficiently dense and that the imaging velocity is accurately known, then we can use the angle decomposition discussed in this paper to evaluate amplitude variation with azimuth and reflection angles. However, we emphasize that this is a relevant exercise only if the reflector illumination is sufficiently dense. Otherwise, AVA effects overlap with illumination effects, rendering the analysis unreliable. Migration velocity analysis in the angle domain may also suffer from the lack of adequate illumination. This partial illumination may deteriorate the moveout which would otherwise be observed in the extended image domain. Furthermore, we do not advocate an implementation of MVA in the angle-domain, but rather in the extended image domain which contains all the relevant information and avoids the additional step of angle decomposition. An extensive discussion of this problem is outside the scope of our paper.

CONCLUSIONS

Angle decomposition based on wavefield extrapolation methods is characterized by robustness in areas with sharp velocity variation and by accuracy in the presence of steeply dipping reflectors. Extended common-image-point gathers constructed at discrete image points provide sufficient information for angle decomposition. The decomposition is based on the planar approximation of the source and receiver wavefields in the immediate vicinity of the image points. Both space-lag and the time-lag extensions are required to completely characterize the reflection geometry given by the local reflection and azimuth angles. However, assuming that information about the reflector slope is available, we could avoid computing one lag of the extended image, usually the vertical. This increases the computational efficiency of the method and makes it affordable for large-scale wide-azimuth imaging projects.
ACKNOWLEDGMENTS

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APPENDIX A

This appendix shows the pseudo-code used for angle decomposition indicating the loop order and the link with the theory discussed in the body of the paper.

<table>
<thead>
<tr>
<th>CIP loop</th>
<th>$c = 1 \ldots N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>read reflector dip</td>
<td>$\mathbf{n}$</td>
</tr>
<tr>
<td>read azimuth reference</td>
<td>$\mathbf{v}$</td>
</tr>
<tr>
<td>rotate azimuth reference</td>
<td>${\mathbf{n}, \mathbf{v}} \rightarrow \mathbf{a}$, (equation 19)</td>
</tr>
<tr>
<td>read velocity</td>
<td>$v$</td>
</tr>
<tr>
<td>read CIP</td>
<td>$R(\lambda, \tau)$</td>
</tr>
<tr>
<td>azimuth angle loop</td>
<td>loop $\phi = 0^\circ \ldots 360^\circ$</td>
</tr>
<tr>
<td>rotate by azimuth</td>
<td>${\mathbf{n}, \phi, \mathbf{a}} \rightarrow \mathbf{q}$, (equation 21)</td>
</tr>
<tr>
<td>reflection angle loop</td>
<td>loop $\theta = 0^\circ \ldots 90^\circ$</td>
</tr>
<tr>
<td>apply slant stack</td>
<td>$R(\lambda, \tau) \Rightarrow R(\phi, \theta)$, (equation 17)</td>
</tr>
<tr>
<td>write</td>
<td>$R(\phi, \theta)$</td>
</tr>
</tbody>
</table>

ANGULAR DECOMPOSITION ALGORITHM FOR EXTENDED CIPs.
Micro-earthquake monitoring with sparsely-sampled data

Paul Savad

ABSTRACT
Micro-seismicity can be used to monitor the migration of fluids during reservoir production and hydro-fracturing operations in brittle formations or for studies of naturally occurring earthquakes in fault zones. Micro-earthquake locations can be inferred using wave-equation imaging under the exploding reflector model, assuming densely sampled data and known velocity. Seismicity is usually monitored with sparse networks of seismic sensors, for example located in boreholes. The sparsity of the sensor network itself degrades the accuracy of the estimated locations, even when the velocity model is accurately known. This constraint limits the resolution at which fluid pathways can be inferred. Wavefields reconstructed in known velocity using data recorded with sparse arrays can be described as having a random character due to the incomplete interference of wave components. Similarly, wavefields reconstructed in unknown velocity using data recorded with dense arrays can be described as having a random character due to the inconsistent interference of wave components. In both cases, the random fluctuations obstruct focusing that occurs at source locations. This situation can be improved using interferometry in the imaging process. Reverse-time imaging with an interferometric imaging condition attenuates random fluctuations, thus producing crisper images which support the process of robust automatic micro-earthquake location. The similarity of random wavefield fluctuations due to model fluctuations and sparse acquisition are illustrated in this paper with a realistic synthetic example.

INTRODUCTION
Seismic imaging based on the single scattering assumption, also known as Born approximation, consists of two main steps: wavefield reconstruction which serves the purpose of propagating recorded data from the acquisition surface back into the subsurface, followed by an imaging condition which serves the purpose of highlighting locations where scattering occurs. This framework holds both when the source of seismic waves is located in the subsurface and the imaging target consists of locating this source, as well as when the source of seismic waves is located on the acquisition surface and the imaging target consists of locating the places in the subsurface where

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scattering or reflection occurs. In this paper, I concentrate on the case of imaging seismic sources located in the subsurface, although the methodology discussed here applies equally well for the more conventional imaging with artificial sources.

An example of seismic source located in the subsurface is represented by micro-earthquakes triggered by natural causes or by fluid injection during reservoir production or fracturing. One application of micro-earthquake location is monitoring of fluid injection in brittle reservoirs when micro-earthquake evolution in time correlates with fluid movement in reservoir formations. Micro-earthquakes can be located using several methods including double-difference algorithms (Waldhauser and Ellsworth 2000), Gaussian-beam migration (Rentsch et al. 2004, 2007), diffraction stacking (Gajewski et al. 2007) or time-reverse imaging (Gajewski and Tessmer 2005; Artman et al., 2010).

Micro-earthquake location using time-reverse imaging, which is also the technique advocated in this paper, follows the same general pattern mentioned in the preceding paragraph: wavefield-reconstruction backward in time followed by an imaging condition extracting the image, i.e. the location of the source. The main difficulty with this procedure is that the onset of the micro-earthquake is unknown, i.e. time $t = 0$ is unknown, so the imaging condition cannot be simply applied as it is usually done in zero-offset migration. Instead, an automatic search needs to be performed in the back-propagated wavefield to identify the locations where wavefield energy focuses. This process is difficult and often ambiguous since false focusing locations might overlap with locations of wavefield focusing. This is particularly true when imaging using an approximate model which does not explain all random fluctuations observed in the recorded data. This problem is further complicated if the acquisition array is sparse, e.g. when receivers are located in a borehole. In this case, the sparsity of the array itself leads to artifacts in the reconstructed wavefield which makes the automatic picking of focused events even harder.

The process by which sampling artifacts are generated is explained in Figures 1(a)-1(d). Each segment in Figure 1(a) corresponds to a wavefront reconstructed from a receiver. For dense, uniform and wide-aperture receiver coverage and for reconstruction using accurate velocity, the wavefronts overlap at the source position, Figure 1(b). This idealized situation resembles the coverage typical for medical imaging, although the physical processes used are different. However, if the velocity used for wavefield reconstruction is inaccurate, then the wavefronts do not all overlap at the source position, Figure 1(c), thus leading to imaging artifacts. Likewise, if receiver sampling is sparse, reconstruction at the source position is incomplete, Figure 1(d), even if the velocity used for reconstruction is accurate. The cartoons depicted in Figures 1(a)-1(d) represent an ideal situation with receivers surrounding the seismic source, which is not typical for seismic experiments. In those cases, source illumination is limited to a range which correlates with the receiver coordinates.

In general, artifacts caused by unknown velocity fluctuations and receiver sampling overlap and, although the two phenomena are not equivalent, their effect on the reconstructed wavefields are analogous. As illustrated in the following sections,
Figure 1: Schematic representation of focus constructions using time reversal. Each line in the plots represents a wavefront reconstructed at the source from a given receiver. The panels represent the following cases: (a) dense acquisition, complete angular coverage and correct velocity, (b) dense acquisition, partial angular coverage and correct velocity, (c) dense acquisition, partial angular coverage and incorrect velocity, and (d) sparse acquisition, partial angular coverage and incorrect velocity. Panel (d) represents the worst case scenario for micro-earthquake imaging.
the general character of those artifacts is that of random wavefield fluctuations. Ideally, the imaging procedure should attenuate those random wavefield fluctuations irrespective of their cause in order to support automatic source identification.

CONVENTIONAL IMAGING CONDITION

Assuming data $D(x,t)$ acquired at coordinates $x$ function of time $t$ (e.g. in a borehole) we can reconstruct the wavefield $V(x,y,t)$ at coordinates $y$ in the imaging volume using an appropriate Green’s function $G(x,y,t)$ corresponding to the locations $x$ and $y$ (Figure 2).

$$V(x,y,t) = D(x,t) *_{t} G(x,y,t) ,$$

where the symbol $*_{t}$ indicates time convolution. The total wavefield $U(y,t)$ at coordinates $y$ due to data recorded at all receivers located at coordinates $x$ is represented by the superposition of the reconstructed wavefields $V(x,y,t)$:

$$U(y,t) = \int x V(x,y,t).$$

A conventional imaging condition (CIC) applied to this reconstructed wavefield extracts the image $R_{CIC}(y)$ as the wavefield at time $t = 0$.

$$R_{CIC}(y) = U(y,t = 0).$$

This imaging procedure succeeds if several assumptions are fulfilled: first, the velocity model used for imaging has to be accurate; second, the numeric solution to the wave-equation used for wavefield reconstruction has to be accurate; third, the data need to be sampled densely and uniformly on the acquisition surface. In this paper, I assume that the first and third assumptions are not fulfilled. In these cases, the imaging is not accurate because contributions to the reconstructed wavefield from the receiver coordinates do not interfere constructively, thus leading to imaging artifacts. As indicated earlier, this situation is analogous to the case of imaging with an inaccurate velocity model, e.g. imaging with a smooth velocity of data corresponding to geology characterized by rapid velocity variations.

Different image processing procedures can be employed to reduce the random wavefield fluctuations. The procedure advocated in this paper uses interferometry for noise cancellation. Interferometric procedures can be formulated in various frameworks, e.g. coherent interferometric imaging (Borcea et al. 2006) or wave-equation migration with an interferometric imaging condition (Sava and Poliannikov 2008).

INTERFEROMETRIC IMAGING CONDITION

Migration with an interferometric imaging condition (IIC) uses the same generic framework as the one used for the conventional imaging condition, i.e. wavefield
reconstruction followed by an imaging condition. However, the difference is that
the imaging condition is not applied to the reconstructed wavefield directly, but it is
applied to the wavefield which has been transformed using pseudo Wigner distribution
functions (WDF) \(Wigner, 1932\). By definition, the zero frequency pseudo WDF of
the reconstructed wavefield \(U(y, t)\) is

\[
W(y, t) = \int_{|t_h| \leq T} \int_{|y_h| \leq Y} U\left(y - \frac{y_h}{2}, t - \frac{t_h}{2}\right) U\left(y + \frac{y_h}{2}, t + \frac{t_h}{2}\right) dt_h dy_h,
\]

where \(Y\) and \(T\) denote averaging windows in space and time, respectively. In general,
\(Y\) is three dimensional and \(T\) is one dimensional. Then, the image \(R_{IIIC}(y)\) is obtained
by extracting the time \(t = 0\) from the pseudo WDF, \(W(y, t)\), of the wavefield \(U(y, t)\):

\[
R_{IIIC}(y) = W(y, t = 0).
\]

The interferometric imaging condition represented by equations 4 and 5 effectively
reduces the artifacts caused by the random fluctuations in the wavefield by filtering
out its rapidly varying components \(Sava and Poliannikov, 2008\). In this paper,
I use this imaging condition to attenuate noise caused by sparse data sampling or
noise caused by random velocity variations. As suggested earlier, the interferometric
imaging condition attenuates both types of noise at once, since it does not explicitly
distinguish between the various causes of random fluctuations.

The parameters \(Y\) and \(T\) defining the local window of the pseudo WDF are selected according to two criteria \(Cohen, 1995\). First, the windows have to be large
enough to enclose a representative portion of the wavefield which captures the random
fluctuation of the wavefield. Second, the window has to be small enough to limit
the possibility of cross-talk between various events present in the wavefield. Furthermore,
cross-talk can be attenuated by selecting windows with different shapes, for
example Gaussian or exponentially-decaying. Therefore, we could in principle define the transformation in equation 4 more generally as

\[ W(y, t) = \int dt h \int d\mathbf{y} W_{\mathbf{Y}}(\mathbf{y}, y) U \left( \mathbf{y} - \frac{y_h}{2}, t - \frac{t_h}{2} \right) U \left( \mathbf{y} + \frac{y_h}{2}, t + \frac{t_h}{2} \right), \]

where \( W_T \) and \( W_Y \) are weighting functions which could represent Gaussian, boxcar or any other local functions (Artman 2011, personal communication). For simplicity, in all examples presented in this paper, the space and time windows are rectangular with no tapering and the size is selected assuming that micro-earthquakes occur sufficiently sparse, i.e. the various sources are located at least twice as far in space and time relative to the wavenumber and frequency of the considered seismic event. Typical window sizes used here are 11 grid points in space and 5 grid points in time.

**EXAMPLE**

I exemplify the interferometric imaging condition method with a synthetic example simulating the acquisition geometry of the passive seismic experiment performed at the San Andreas Fault Observatory at Depth (Chavarria et al., 2003; Vasconcelos et al., 2008). This numeric experiment simulates waves propagating from three micro-earthquake sources located in the fault zone, Figure 3, which are recorded in a deviated well located at a distance from the fault. For the imaging procedure described in this paper, the micro-earthquakes represent the seismic sources. This experiment uses acoustic waves, corresponding to the situation in which we use the P-wave mode recorded by the three-component receivers located in the borehole, Figures 4(b) and 5(b). The three sources are triggered 40 ms apart and the triggering time of the second source is conventionally taken to represent the origin of the time axis.

The goal of this experiment is to locate the source positions by focusing data recorded using dense acquisition in media with random fluctuations or by focusing data recorded using sparse acquisition arrays in media without random fluctuations. In the first case, the imaging artifacts are caused by the fact that data are imaged with a velocity model that does not incorporate all random fluctuations of the model used for data simulation, while in the second case, the imaging artifacts are caused by the fact that the data are sampled sparsely in the borehole array. The third case is a combination of acquisition with two sparse arrays, and imaging with an inaccurate velocity model.

Figures 7(a)-7(b), 8(a)-8(b) and 9(a)-9(b) show the wavefields reconstructed in reverse time around the target location. From left to right, the panels represent the wavefield at different times. As indicated earlier, the time at which source 2 focuses is selected as time \( t = 0 \), although this convention is not relevant for the experiment and any other time could be selected as reference. The experiment depicted in Figures 7(a)-7(b) corresponds to modeling in a model with random fluctuations and migration in a smooth background model. In this experiment, the data used for
Figure 3: Geometry of the sources used in the numeric experiment. The horizontal and vertical separation between sources is 250 m. The sources are triggered with 40 ms delays in the order indicated by their numbers. Time $t = 0$ is conventionally set to the the triggering moment of source 2.

imaging are densely-sampled in the borehole, i.e. there are 81 receivers separated by approximately 12 m. In contrast, the experiment depicted in 8(a)-8(b) corresponds to modeling and migration in the smooth background model. In this experiment, the data are sparsely-sampled in the borehole, i.e. there are only 6 receivers obtained by selecting every 16th receiver from the original set. In all cases, panels (a) correspond to imaging with a conventional imaging condition, i.e. simply select the reconstructed wavefield at various times, and panels (b) correspond to imaging with the interferometric imaging condition, i.e. select various times from the wavefield transformed with a pseudo-WDF of 11 grid points in space and 5 grid points in time. For this example, WDF window corresponds to 44 m in space and 2 ms in time.

Figure 7(a) shows significant random fluctuations caused by wavefield reconstruction using an inaccurate velocity model. The fluctuations caused by the random velocity and encoded in the recorded data are not corrected during wavefield reconstruction and they remain present in the model. Likewise, Figure 8(a) shows significant random fluctuations caused by reconstruction using the sparse borehole data. However, the pseudo WDF applied to the reconstructed wavefields attenuates the rapid wavefield fluctuations and leads to sparser, better focused images that are easier to use for source location. This conclusion applies equally well for the experiments depicted in Figures 7(a)-7(b) or 8(a)-8(b).

The final example corresponds to the case of acquisition with two separate sparse arrays, Figures 6(a)-6(b). As expected, the wavefields are far less noisy after the application of the WDF, and the focusing is increased due to the larger array aperture. This facilitates an automatic procedure for focusing identification, since most of the spurious noisy is eliminated from the image.

Finally, I note that the 2D imaging results from this example show better focusing than what would be expected in 3D. This is simply because the 1D acquisition in the borehole cannot constrain the 3D location of the micro-earthquakes, i.e. the azimuthal resolution is poor, especially if scatterers are not present in the model used for imaging. This situation can be improved by using data acquired in several boreholes
Figure 4: (a) Wavefields simulated in random media and (b) data acquired with a dense receiver array. Overlaid on the model and wavefield are the positions of the sources and borehole receivers. The boxed area corresponds to the images depicted in Figures 7(a)–7(b).
Figure 5: (a) Wavefields simulated in smooth media and (b) data acquired with a sparse receiver array. Overlain on the model and wavefield are the positions of the sources and borehole receivers. The boxed area corresponds to the images depicted in Figures 8(a)-8(b).
Figure 6: (a) Wavefields simulated in random media and (b) data acquired with two sparse receiver arrays. Overlaid on the model and wavefield are the positions of the sources and borehole receivers. The boxed area corresponds to the images depicted in Figures 9(a) and 9(b). The top traces in panel (b) correspond to the vertical array, and the other traces correspond to the sparse deviated array.
Figure 7: Images corresponding to migration of the densely-sampled data (Figure 4(b)) modeled in the random velocity by (a) conventional I.C. and (b) interferometric I.C. using the background velocity. The left-most panel shows focusing at source 1, the middle panel shows focusing at source 2, and the right-most panel shows focusing at source 3. The overlain dots represent the exact source positions.
Figure 8: Images corresponding to migration of the sparsely-sampled data (Figure 5(b)) modeled in the background velocity by (a) conventional I.C. and (b) interferometric I.C. using the background velocity. The left-most panel shows focusing at source 1, the middle panel shows focusing at source 2, and the right-most panel shows focusing at source 3. The overlain dots represent the exact source positions.
Figure 9: Images corresponding to migration of the dual sparsely-sampled data (Figure 6(b)) modeled in the random velocity by (a) conventional I.C. and (b) interferometric I.C. using the background velocity. The left-most panel shows focusing at source 1, the middle panel shows focusing at source 2, and the right-most panel shows focusing at source 3. The overlain dots represent the exact source positions.
or by using additional information extracted from the wavefields, e.g. polarization of multicomponent data.

**CONCLUSIONS**

The interferometric imaging condition used in conjunction with time-reverse imaging reduces the artifacts caused by random velocity fluctuations that are unaccounted-for in imaging and by the sparse wavefield sampling on the acquisition array. The images produced by this procedure are crisper and support automatic picking of micro-earthquake locations. Imaging with sparse arrays allows increased aperture for identical acquisition cost with that of a narrower but denser array. At the same time, a larger aperture improves focusing of the events, thus facilitating automatic event identification. The interferometric imaging procedure has a similar structure to conventional imaging and the moderate cost increase is proportional to the size of the windows used by the pseudo Wigner distribution functions. The source positions obtained using this procedure can be used to monitor fluid injection or for studies of naturally occurring earthquakes in fault zones.

**ACKNOWLEDGMENT**

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